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PE | Industrial and Systems

Reference Handbook
Version 1.1

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Preface

About the Handbook

The Principles and Practice of Engineering (PE) Industrial and Systems exam is computer-based, and NCEES will supply all the resource material that you may use during the exam. Reviewing the *PE Industrial and Systems Reference Handbook* before exam day will help you become familiar with the charts, formulas, tables, and other reference information provided. You will not be allowed to bring your personal copy of the *PE Industrial and Systems Reference Handbook* into the exam room. Instead, the computer-based exam will include a PDF version of the handbook for your use. No printed copies of the handbook will be allowed in the exam room.

The PDF version of the *PE Industrial and Systems Reference Handbook* that you use on exam day will be very similar to this one. However, pages not needed to solve exam questions—such as the cover and introductory material—may not be included in the exam version. In addition, NCEES will periodically revise and update the handbook, and each PE Industrial and Systems exam will be administered using the updated version.

The *PE Industrial and Systems Reference Handbook* does not contain all the information required to answer every question on the exam. Theories, conversions, formulas, and definitions that examinees are expected to know have not been included. The handbook is intended solely for use on the NCEES PE Industrial and Systems exam.

Other Supplied Exam Material

In addition to the *PE Industrial and Systems Reference Handbook*, the exam will include codes and standards for your use. A list of the material that will be included in your exam is available at ncees.org along with the exam specifications. Any additional material required for the solution of a particular exam question will be included in the question itself. You will not be allowed to bring personal copies of any material into the exam room.

Updates on Exam Content and Procedures

NCEES.org is our home on the web. Visit us there for updates on everything exam-related, including specifications, exam-day policies, scoring, and practice tests.

Errata

To report errata in this book, log in to your MyNCEES account and send a message. Examinees are not penalized for any errors in the handbook that affect an exam question.

Contributors

The *PE Industrial and Systems Reference Handbook* was developed by members of the Institute of Industrial and Systems Engineers.

Contents

1 Systems Engineering

Modeling Techniques

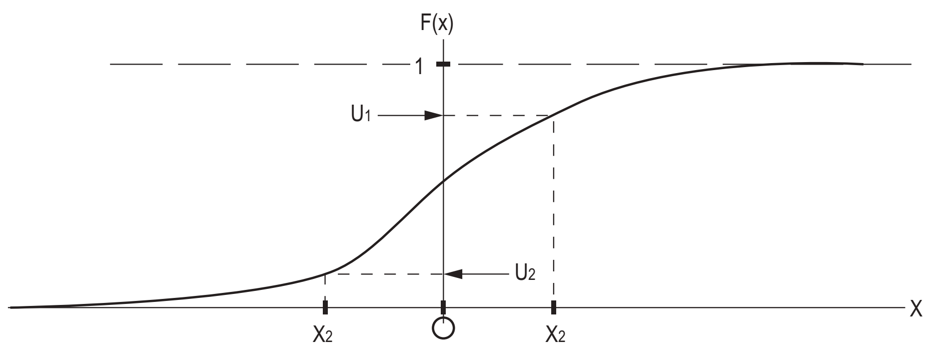
Inverse Transform Method

X is a continuous random variable with cumulative distribution $F(x)$.

U_i is a uniform random number between 0 and 1.

The value of X_i corresponding to U_i can be calculated by solving U_i = $F(X_i)$ for X_j.

The solution obtained is $X_i = F^{-1}(U_j)$
where F^{-1} is the inverse function of F



Inverse Transform Method for Continuous Random Variables

Linear Programming

General Linear Programming

The general linear programming (LP) problem is:

Maximize $Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$

Subject to:

$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1$

$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1$

. . .

. . .

$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m$

where

$x_1, \ldots, x_n \geq 0$

Dual Linear Programming

Primal	Dual
Maximize $Z = cx$	Minimize $W = yb$
Subject to:	Subject to:
$Ax \leq b$	$yA \geq c$
$x \geq 0$	$y \geq 0$

Assume A is a matrix of size $[m\ x\ n]$, then:

y is an $[1\ x\ m]$ vector
c is an $[1\ x\ n]$ vector
b is an $[m\ x\ 1]$ vector
x is an $[n\ x\ 1]$ vector

Northwest Corner Rule

Steps in the algorithm:

1. Starting from the northwest (upper left-hand) corner of the transportation table, allocate (within the supply constraint of Source 1 and the demand constraint of Destination 1) as much quantity as possible to cell (1,1) from Origin 1 to Destination 1.
    a. The first allocation will satisfy either the supply capacity of Source 1 or the destination requirement of Destination 1.
    b. If the demand requirement for Destination 1 is satisfied but the supply capacity for Source 1 is not exhausted, move horizontally to the next cell (1,2) for the next allocation.
    c. If the demand requirement for Destination 1 is not satisfied but the supply capacity for Source 1 is exhausted, move down to cell (2,1).
    d. If the demand requirement for Destination 1 is satisfied and the supply capacity for Source 1 is also exhausted, move on to cell (2,2).
2. Continue the allocation in the same manner toward the southeast corner of the transportation table until the supply capacities of all sources are exhausted and the demands of all destinations are satisfied.

Queueing Models

Definitions

P_n = probability of *n* units in system

L = expected number of units in the system

L_q = expected number of units in the queue

W = expected waiting time in system

W_q = expected waiting time in queue

λ = mean arrival rate

$\tilde{\lambda}$ = effective arrival rate

μ = mean service rate

ρ = server utilization factor

s = number of servers

Kendall notation for describing a queueing system: *A* / *B* / *s* / *M*

where

A = arrival process

B = service time distribution

s = number of servers

M = total number of customers including those in service and in queue

Fundamental Relationships

$L = \lambda W$; $L_q = \lambda W_q$ (Little's Law)

$$W = W_q + \frac{1}{\mu}$$

$$\rho = \frac{\lambda}{s\mu}$$

Single Server Models (S = 1)

Poisson Input—Exponential Service Time: *M* = ∞

$$P_0 = 1 - \frac{\lambda}{\mu} = 1 - \rho$$

$$P_n = (1 - \rho)\rho^n = P_0\rho^n$$

$$L = \frac{\rho}{1 - \rho} = \frac{\lambda}{\mu - \lambda}$$

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)}$$

$$W = \frac{1}{\mu(1 - \rho)} = \frac{1}{\mu - \lambda}$$

$$W_q = W - \frac{1}{\mu} = \frac{\lambda}{\mu(\mu - \lambda)}$$

Finite Queue: *M* < ∞

$$\tilde{\lambda} = \lambda(1 - P_n)$$

$$P_0 = \frac{(1 - \rho)}{(1 - \rho^{M+1})}$$

$$Pn = \left(\frac{1 - \rho}{1 - \rho^{M+1}}\right)\rho^n$$

$$L = \frac{\rho}{1 - \rho} - \frac{(M + 1)\rho^{M+1}}{1 - \rho^{M+1}}$$

$$L_q = L - (1 - P_0)$$

Poisson Input—Arbitrary Service Time: Variance σ^2 Known. For Constant Service Time, $\sigma^2 = 0$

$$P_0 = 1 - \rho$$

$$L_q = \frac{\lambda^2\sigma^2 + \rho^2}{2\,(1 - \rho)}$$

$$L = \rho + L_q$$

$$W_q = \frac{L_q}{\lambda}$$

$$W = W_q + \frac{1}{\mu}$$

Poisson Input—Erlang Service Times, $\sigma^2 = 1/\left(ku^2\right)$

$$L_q = \left(\frac{1 + k}{2k}\right)\left(\frac{\lambda^2}{\mu\,(\mu - \lambda)}\right) = \frac{\left(\frac{\lambda^2}{k\mu^2} + \rho^2\right)}{2(1-\rho)}$$

$$W_q = \left(\frac{1 + k}{2k}\right)\left(\frac{\lambda}{\mu\,(\mu - \lambda)}\right)$$

$$W = W_q + \frac{1}{\mu}$$

Multiple Server Model (s > 1)

Poisson Input—Exponential Service Times

$$P_0 = \left[\sum_{n=0}^{s-1} \frac{\left(\frac{\lambda}{\mu}\right)^n}{n!} + \frac{\left(\frac{\lambda}{\mu}\right)^s}{s!}\left(\frac{1}{1-\frac{\lambda}{s\mu}}\right)\right] = \frac{1}{\sum_{n=0}^{s-1}\frac{(s\rho)^n}{n!} + \frac{(s\rho)^s}{s!(1-\rho)}}$$

$$L_q = \frac{P_0\left(\frac{\lambda}{\mu}\right)^s \rho}{s!(1-\rho)^2} = \frac{P_0 s^s \rho^{s+1}}{s!(1-\rho)^2}$$

$$P_n = \frac{P_0\left(\frac{\lambda}{\mu}\right)^n}{n!} \qquad 0 \le n \le s$$

$$P_n = \frac{P_0\left(\frac{\lambda}{\mu}\right)^n}{s!\,s^{n-s}} \qquad n \ge s$$

$$W_q = \frac{L_q}{\lambda}$$

$$W = W_q + \frac{1}{\mu}$$

$$L = L_q + \frac{\lambda}{\mu}$$

Calculations for P_0 and L_q can be time consuming. The following table gives equations for 1, 2, and 3 servers.

1, 2, and 3 Server Queuing Equations

s	P_0	L_q
1	$1-\rho$	$\frac{\rho^2}{1-\rho}$
2	$\frac{1-\rho}{1+\rho}$	$\frac{2\rho^3}{1-\rho^2}$
3	$\frac{2(1-\rho)}{2+4\rho+3\rho^2}$	$\frac{9\rho^4}{2+2\rho-\rho^2-3\rho^3}$

Waiting Time Systems

Utilization $u = \frac{p}{a \times m}$

where

m = number of servers

p = processing time

a = interarrival time

$$T_q = \left(\frac{\text{processing time}}{m} \right) \times \left(\frac{\text{utilization}^{\sqrt{2(m+1)}-1}}{1-\text{utilization}} \right) \times \left(\frac{CV_a^2 + CV_p^2}{2} \right)$$

where

CV_a = coefficient of variation for interarrivals

CV_p = coefficient of variation of processing time

Flow time $T = T_q + p$

Inventory in service $I_p = m \times u$

Inventory in the queue $I_q = T_q/a$

Inventory in the system $I = I_p + I_q$

Project Management and Planning

Nomenclature

BCWS = Budgeted cost of work scheduled (Planned Value)

ACWP = Actual cost of work performed (Actual Cost)

BCWP = Budgeted cost of work performed (Earned Value)

CV = Cost variance

SV = Schedule variance

CPI = Cost Performance Index

SPI = Schedule Performance Index

ETC = Estimate to Complete

EAC = Estimate at Completion (forecast of the project's final cost)

BAC = Budget at Completion (the original project estimate)

VAC = Variance at Completion

Variances

CV = BCWP – ACWP

SV = BCWP – BCWS

Indices

$\text{CPI} = \frac{\text{BCWP}}{\text{ACWP}}$

$\text{SPI} = \frac{\text{BCWP}}{\text{BCWS}}$

Forecasting

$\text{ETC} = \frac{\text{BAC} - \text{BCWP}}{\text{CPI}} = \text{EAC} - \text{ACWP}$

$\text{EAC} = \frac{\text{BAC}}{\text{CPI}} = \text{ACWP} + \text{ETC}$

VAC = BAC – EAC

Critical Path Method

d_{ij} = duration of activity (i, j)

CP = critical path (longest path)

T = duration of project = $\sum_{(i,j)\epsilon CP} d_{ij}$

Project Evaluation Review Technique (PERT)

(a_{ij}, b_{ij}, c_{ij}) = (optimistic, most likely, pessimistic) durations for activity (i, j)

μ_{ij} = mean duration of activity (i, j)

σ_{ij} = standard deviation of the duration of activity (i, j)

μ = project mean duration

σ = standard deviation of project duration

$$\mu_{ij} = \frac{a_{ij} + 4b_{ij} + c_{ij}}{6}$$

$$\sigma_{ij} = \frac{c_{ij} - a_{ij}}{6}$$

$$\mu = \sum_{(ij)\epsilon CP} \mu_{ij}$$

$$\sigma^2 = \sum_{(ij)\epsilon CP} \sigma_{ij}^2$$

2 Facilities Engineering and Planning

Network Optimization

Minimal Spanning Tree Algorithm

1. Start with any node in the unconnected set and add to the connected set.
2. Select a node in the unconnected set that yields the shortest branch to a node in the connected set.
   a. Link the selected node to the connected set via the identified shortest branch.
3. If any unconnected nodes remain, repeat Step 2.

Dijkstra's Algorithm

Let u_i be the shortest distance from source node 1 to node i
d_{ij} is the length of $arc(i, j)$
The label for node j is defined as $[u_j, i] = [u_i + d_{ij}, i]$

1. Select a source node, i.
2. Compute the temporary labels for each node j that can be reached from node i, provided j is not permanently labeled.
3. If node j is already labeled through another node, but new temporary label would be less, replace the previous temporary label with the new label.
4. Select the temporary route, r, with the shortest distance of all the temporary routes.
   a. Set this to a permanent assignment.
   b. Set i to r.
5. If all nodes are permanently assigned, stop. Otherwise, return to Step 2.

Layout Design Techniques

Relationship Chart

Value	Closeness	Code	Reason for Closeness
A	Absolutely necessary	1	Frequency of use
E	Especially important	2	Degree of personal contact
I	Important	3	Use of common equipment
O	Ordinary closeness ok	4	Share same space
U	Unimportant	5	Use common records
X	Undesirable	6	Information flow

Lighting Design

Lighting Design Procedure

1. Determine level of illumination
2. Determine the room cavity ratio (RCR)
3. Determine the ceiling cavity ratio (CCR)
4. Determine the wall reflections (WR) and the effective ceiling reflectance (ECR)
5. Determine the coefficient of utilization (CU)
6. Determine the light loss factor (LLF)
7. Calculate the number of lamps and luminaries
8. Determine the location of the luminaries

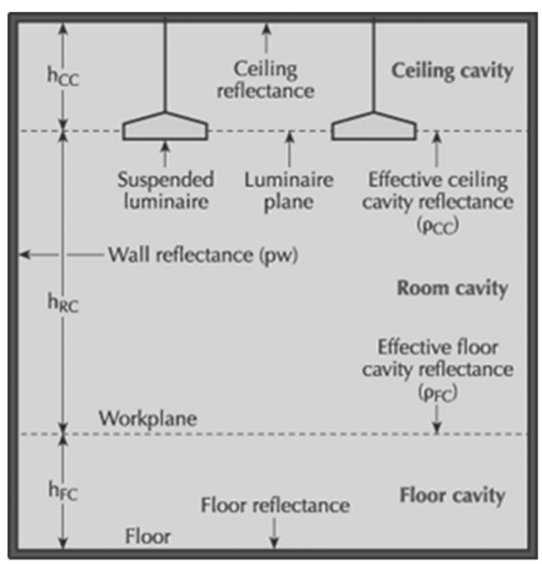
Room Cavity Ratio (RCR)

$$\text{RCR} = \frac{(5)\begin{pmatrix}\text{Height from the working}\\ \text{surface to the luminaries}\end{pmatrix}\begin{pmatrix}\text{Room}\\ \text{length}\end{pmatrix} + \begin{pmatrix}\text{Room}\\ \text{width}\end{pmatrix}}{\begin{pmatrix}\text{Room}\\ \text{length}\end{pmatrix}\begin{pmatrix}\text{Room}\\ \text{width}\end{pmatrix}}$$

Ceiling Cavity Ratio (CCR)

$$\text{CCR} = \frac{\begin{pmatrix}\text{Height from}\\ \text{luminaries to ceiling}\end{pmatrix}(\text{RCR})}{\begin{matrix}\text{Height from the working}\\ \text{surface to the luminaries}\end{matrix}}$$

$$\text{Number of lamps} = \frac{\begin{pmatrix}\text{Required level}\\ \text{of illumination}\end{pmatrix}\begin{pmatrix}\text{Area to}\\ \text{be lit}\end{pmatrix}}{(\text{CU})(\text{LLF})\begin{pmatrix}\text{Lamp output at}\\ \text{70\% of rated life}\end{pmatrix}}$$



Luminaries Variables

$$\begin{matrix}\text{Number}\\ \text{of}\\ \text{luminaries}\end{matrix} = \frac{\begin{pmatrix}\text{Number}\\ \text{of}\\ \text{lamps}\end{pmatrix}}{\begin{pmatrix}\text{Lamps}\\ \text{per}\\ \text{luminary}\end{pmatrix}}$$

Luminaries Cleaning Frequency

	Dirt—Condition[a]																			
	Clean—Offices, Light Assembly, or Inspection					Medium—Mill Offices Paper Processing or Light Machining					Dirty—Heat Treating, High-Speed Printing, or Medium Machining					Very Dirty—Foundry or Heavy Machining				
	Months between Cleaning					Months between Cleaning					Months between Cleaning					Months between Cleaning				
Luminaries	6	12	24	36	48	6	12	24	36	48	6	12	24	36	48	6	12	24	36	48
Filament reflector lamps	0.95	0.93	0.89	0.86	0.83	0.94	0.89	0.85	0.81	0.78	0.87	0.84	0.79	0.74	0.70	0.83	0.74	0.60	0.56	0.52
High-intensity discharge lamps	0.94	0.90	0.84	0.80	0.75	0.92	0.88	0.80	0.74	0.69	0.90	0.83	0.76	0.68	0.64	0.86	0.79	0.69	0.63	0.57
Fluorescent lamps in uncovered fixtures	0.97	0.94	0.89	0.87	0.85	0.93	0.90	0.85	0.83	0.79	0.93	0.87	0.80	0.73	0.70	0.88	0.83	0.75	0.70	0.64
Fluorescent lamps in prismatic lens fixtures	0.92	0.88	0.83	0.80	0.78	0.88	0.84	0.77	0.73	0.71	0.82	0.78	0.71	0.67	0.62	0.78	0.72	0.64	0.60	0.57

% Effective Ceiling Reflectance (ECR) for Various CCR, WR, and BCR Combinations

	BCR																	
	80						65						50					
	WR						WR						WR					
CCR	80	65	50	35	10	5	80	65	50	35	10	5	80	65	50	35	10	5
0.5	76	74	72	69	67	65	64	60	58	56	54	52	49	47	46	44	42	41
1.0	74	71	67	63	57	56	60	55	53	49	45	43	48	45	43	39	36	35
1.5	72	67	62	55	49	47	58	52	49	44	38	36	47	44	40	35	21	28
2.0	69	63	56	49	41	39	55	49	44	38	32	30	46	42	37	31	26	25
2.5	67	60	51	43	35	33	54	45	40	33	26	25	46	40	35	28	22	21
3.0	65	57	47	38	30	28	53	42	38	29	22	21	45	39	32	25	19	18
3.5	63	54	43	34	26	25	52	39	33	26	18	17	44	38	30	23	17	16
4.0	61	52	46	31	22	21	50	37	31	23	15	14	44	38	28	21	15	13
5.0	58	46	35	26	18	15	48	33	26	18	9	8	42	35	25	18	12	10
8.0	50	36	25	17	11	6	41	24	18	11	5	3	40	30	19	13	7	5

	BCR																	
	37						10						5					
	WR						WR						WR					
CCR	80	65	50	35	10	5	80	65	50	35	10	5	80	65	50	35	10	5
0.5	36	34	32	31	29	28	12	12	11	11	9	8	8	8	7	6	5	5
1.0	35	33	31	20	26	25	14	13	12	11	9	8	10	7	8	7	5	4
1.5	35	33	20	26	21	20	16	15	12	11	8	7	14	11	9	7	4	4
2.0	35	32	28	23	18	17	18	17	13	10	8	6	15	12	10	7	4	4
2.5	35	31	26	21	16	13	20	19	13	10	7	5	17	14	10	8	3	3
3.0	35	31	24	20	14	12	21	20	13	10	7	4	19	15	11	8	3	3
3.5	35	31	23	18	12	10	22	21	13	10	7	4	20	16	11	8	3	3
4.0	34	30	23	17	10	8	23	22	14	10	7	3	20	17	12	8	3	2
5.0	34	29	21	16	9	7	25	23	14	10	6	3	23	18	12	8	3	2
8.0	34	28	17	11	5	4	27	24	13	10	5	2	26	19	12	6	3	1

Coefficients of Utilization (CU) for Standard Luminaries

			ECR												
			80%				50%				10%				0%
	Spacing Not		WR				WR				WR				WR
Luminaire	to Exceed	RCR	80%	50%	30%	10%	80%	50%	30%	10%	80%	50%	30%	10%	0%
Filament reflector lamps	1.5 × mounting height	1	1.11	1.09	1.07	1.03	1.04	1.02	1.00	.98	.96	.95	.94	.93	.91
		2	1.04	1.00	.95	.92	.99	.95	.92	.88	.92	.90	.87	.85	.83
		3	.95	.92	.88	.82	.92	.88	.84	.80	.85	.83	.80	.77	.75
		4	.90	.85	.79	.73	.86	.81	.76	.71	.79	.77	.73	.70	.68
		5	.82	.77	.71	.65	.80	.75	.69	.64	.75	.71	.67	.63	.61
		10	.58	.50	.43	.38	.54	.49	.43	.38	.51	.47	.42	.38	.36
High-intensity discharge lamps (mercury, metal halide, or sodium)	1.3 × mounting height	1	.89	.87	.84	.82	.81	.80	.78	.77	.74	.73	.72	.71	.70
		2	.82	.79	.75	.72	.77	.74	.71	.68	.70	.68	.66	.64	.63
		3	.76	.72	.67	.63	.72	.68	.64	.61	.65	.63	.60	.58	.56
		4	.70	.66	.61	.57	.67	.63	.58	.55	.57	.55	.54	.53	.51
		5	.64	.60	.55	.51	.62	.58	.53	.49	.56	.54	.52	.49	.46
		10	.45	.40	.34	.30	.42	.38	.34	.30	.40	.36	.41	.20	.28
Fluorescent lamps in uncovered fixtures	1.3 × mounting height	1	.88	.85	.82	.79	.79	.65	.72	.71	.65	.64	.63	.62	.59
		2	.78	.75	.70	.65	.71	.67	.63	.59	.60	.57	.55	.52	.50
		3	.69	.66	.60	.55	.63	.59	.54	.50	.54	.51	.48	.45	.42
		4	.61	.59	.52	.46	.56	.52	.47	.43	.48	.45	.41	.38	.36
		5	.53	.51	.44	.39	.51	.46	.40	.36	.43	.40	.36	.33	.30
		10	.35	.30	.23	.19	.32	.27	.21	.18	.26	.23	.19	.16	.14
Flourescent lamps in prismatic lens fixtures	1.2 × mounting height	1	.65	.63	.61	.59	.60	.59	.58	.56	.56	.55	.54	.53	.52
		2	.60	.57	.54	.51	.56	.54	.51	.49	.51	.50	.49	.47	.46
		3	.54	.51	.48	.44	.51	.49	.46	.43	.47	.46	.44	.42	.41
		4	.49	.46	.42	.39	.48	.44	.41	.38	.44	.42	.39	.37	.36
		5	.45	.42	.37	.34	.44	.40	.36	.34	.40	.38	.35	.33	.32
		10	.31	.26	.21	.18	.29	.25	.21	.18	.27	.24	.20	.18	.17

Heating and Cooling

Heating Requirement

$$Q_H = Q_F + Q_R + Q_G + Q_D + Q_W + Q_I$$

where

Q_H = total heat loss

Q_F = heat loss through the floor

Q_R = heat loss through the roof

Q_G = heat loss through glass windows

Q_D = heat loss through doors

Q_W = heat loss through walls

Q_I = heat loss due to infiltration

Individual Heat Losses

$$Q = AU\ (t_i - t_o)$$

where

Q = heat loss for facility component

A = area of facility component

U = coefficient of transmission of facility component

t_i = temperature inside the facility

t_o = temperature outside the facility

Cooling Requirement

$$Q_c = Q_F + Q_R + Q_G + Q_D + Q_W + Q_V + Q_s + Q_L + Q_P$$

where

Q_C = total cooling load

Q_F = cooling load due to the floor

Q_R = cooling load due to the roof

Q_G = cooling load due to glass windows

Q_D = cooling load due to doors

Q_W = cooling load due to walls

Q_V = cooling load due to ventilation

Q_S = cooling load due to solar radiation

Q_L = cooling load due to lighting

Q_P = cooling load due to personnel

Cooling Load from Solar Radiation

$$Q_S = A \times H \times S$$

where

A = area of glass surface

H = heat absorption of building surface

S = shade factor for various types of shading

Individual Cooling Loads

$Q = AU\ (t_i - t_o)$

where

Q = cooling load for facility component

A = area of facility component

U = coefficient of transmission of facility component

t_i = temperature inside the facility

t_o = temperature outside the facility

Capacity Analysis

Resource Requirements

$M_j = \sum_{i=1}^{n} \frac{P_{ij} T_{ij}}{C_{ij}}$

where

M_j = number of units of resource j required per production period

P_{ij} = desired production rate for product i on resource j, measured in pieces per production period

T_{ij} = standard time to perform operation j on product i

C_{ij} = number of hours in the production period available for the production of product i on resource j

n = number of products

Machining Formulas

Drilling Material Removal Rate (MRR)

$$Drilling\ MRR = \left(\frac{\pi}{4}\right) D^2 f N$$

where

D = drill diameter

f = feed rate per revolution

N = rpm of the drill

Slab Milling Cutting Speed

$$V = \pi D N$$

where

V = cutting speed, the peripheral speed of the cutter

D = cutter diameter

N = cutter rpm

Slab Milling and Face Milling Feed per Tooth

$$f = \frac{v}{Nn}$$

where

f = feed per tooth

v = workpiece speed

n = number of teeth on the cutter

N = cutter rpm

Slab Milling Cutting Time

$$t = \frac{(l + l_c)}{v}$$

Face Milling Cutting Time

$$t = \frac{(l + 2l_c)}{v}$$

where

t = cutting time

l = length of workpiece

l_c = additional length of cutter’s travel (contact length)

v = workpiece speed

Slab Milling Material Removal Rate (MRR)

If $l_{contact} \ll l$, then

$$Slab\ Milling\ MRR = \frac{lwd}{t}$$

where

l = length of workpiece

w = minimum (width of the cut, length of the cutter)

d = depth of cut

t = cutting time

Face Milling Material Removal Rate

$$Face\ Milling\ MRR = wdv$$

where

w = width

d = depth of cut

v = workpiece speed

Turning Surface Cutting Speed

$$v = \pi D_{\text{avg}} N$$

where

D_{avg} = average diameter of workpiece

N = workplace rotation speed rpm

Turning Material Removal Rate

$$Turning\ MRR = \pi D_{\text{avg}} d f N$$

where

f = feed per revolution

d = depth of cut

D_{avg} = average diameter of workpiece

N = workpiece rotational speed rpm

Turning Cutting Time

$$t = \frac{l}{fN}$$

where

f = feed per revolution

l = length of cut

N = workpiece rotational speed rpm

Taylor Tool Life Formula

$$VT^n = C$$

where

V = cutting speed

T = time before the tool reaches a certain percentage of possible wear

C, n = constants that depend on the material and on the tool

Site Selection Methods

Rectilinear Facility Location Problem

Single Facility Minisum Location Formula

$$Minimize\ f(X) = \sum_{i=1}^{m} w_i|x - a_i| + \sum_{i=1}^{m} w_i\ |y - b_i|$$

where

$f(X)$ = annual cost of travel between new facility X and existing facility i

w_i = proportionality constant

$x = x$ location of the new facility

$a_i = x$ location of existing facility i

$y = y$ location of the new facility

$b_i = y$ location of existing facility i

m = total existing facilities

Single Facility Minimax Location Formula

$$Minimize\ f(X) = maximum[(|x - a_i| + |y - b_i|), i = 1,2, \dots m]$$

where

$f(X)$ = annual cost of travel between new facility X and existing facility i

$x = x$ location of the new facility

$a_i = x$ location of existing facility i

$y = y$ location of the new facility

$b_i = y$ location of existing facility i

m = total existing facilities

Plant Location Minimal Cost

The following is one formulation of a discrete plant location problem.

$$Minimize\ z = \sum_{i=1}^{m}\sum_{j=1}^{n} C_{ij}y_{ij} + \sum_{j=1}^{n} f_j x_j$$

Subject to:

$$\sum_{i=1}^{m} y_{ij} \le m x_j, \quad j = 1, \ldots, n$$

$$\sum_{j=1}^{n} y_{ij} = 1, \quad i = 1, \ldots, m$$

$$y_{ij} \ge 0, \quad for\ all\ i, j$$

$$x_j = (0, 1),\ for\ all\ j$$

where

m = number of customers

n = number of possible plant sites

y_{ij} = portion of the demand of customer i which is satisfied by a plant located at site j

$i = 1, \ldots, m$

$j = 1, \ldots, n$

x_j = 1, if a plant is located at site j; 0, otherwise

c_{ij} = cost of supplying the entire demand of customer i from a plant located at site j

f_j = fixed cost resulting from locating a plant at site j

Material Handling Techniques and Equipment

Euclidean Distance

$$D = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Rectilinear (or Manhattan) Distance

$$D = |x_1 - x_2| + |y_1 - y_2|$$

Chebyshev (Simultaneous x and y Movement)

$$D = max(|x_1 - x_2|, |y_1 - y_2|)$$

Crane-Based Systems Calculations

Loads Stacked per Bay

$$Loads\ Stacked\ per\ Bay = \frac{height\ of\ storage\ building}{load\ height + clearance\ per\ stack} - 1$$

OR

$$Loads\ Stacked\ per\ Bay = \frac{height\ of\ storage\ building - (ceiling\ clearance + floor\ clearance)}{load\ height + clearance\ per\ stack}$$

Bays per Row

$$Bays\ per\ Row = \frac{number\ of\ units\ to\ be\ stored}{(number\ of\ rows) \times (number\ of\ loads\ vertically\ stacked\ per\ row)}$$

System Width

$$System\ Width = (aisle\ unit\ \times\ number\ of\ cranes)$$

Note: Aisle unit should be 3 x Length + 2 feet for clearance.

System Length

$$System\ Length = (width\ of\ bay + clearance) \times (bays\ per\ row) + (crane\ clearance)$$

Note: Crane clearance is usually 25 feet.

3 Operations Engineering

Forecasting Methods

Moving Average

$$\hat{d}_t = \frac{\sum_{i=1}^{n} d_{t-i}}{n}$$

where

$\hat{d}_t$ = forecasted demand for period t

d_{t-i} = actual demand for ith period preceding t

n = number of time periods to include in the moving average

Exponentially Weighted Moving Average or Exponential Smoothing Method

$$\hat{d}_t = \alpha d_{t-1} + (1-\alpha)\hat{d}_{t-1}$$

where

$\hat{d}_t$ = forecasted demand for period t

d_t = actual demand for period t

α = smoothing constant, $0 \le \alpha \le 1$

Exponential Smoothing with Trend Adjustment or Holt's Method

$F_t = \alpha$ (actual demand last period) + $(1-\alpha)$(forecast last period + trend estimate last period)

or

$F_t = \alpha(A_{t-1}) + (1-\alpha)(F_{t-1} + T_{t-1})$

$T_t = \beta$(forecast this period – forecast last period) + $(1-\beta)$(trend estimate last period)

or

$T_t = \beta(F_t - F_{t-1}) + (1-\beta)T_{t-1}$

where

F_t = exponentially smoothed forecast of the data series in period t

T_t = exponentially smoothed trend in period t

A_t = actual demand in period t

α = smoothing constant for the average $(0 \le \alpha \le 1)$

β = smoothing constant for the trend $(0 \le \beta \le 1)$

Forecast including trend

$FIT_t = F_t + T_t$

Holt-Winters' Method of Forecasting

The basic equations for their method are given by:

$$S_t = \alpha \frac{y_t}{I_{t-L}} + (1-\alpha)(S_{t-1} + b_{t-1})$$ Overall Smoothing

$$b_t = \gamma(S_t - S_{t-1}) + (1-\gamma) b_{t-1}$$ Trend Smoothing

$$I_t = \beta \frac{y_t}{S_t} + (1-\beta) I_{t-L}$$ Seasonal Smoothing

$$F_{t+m} = (S_t + m b_i) I_{t-L+m}$$ Forecast

where

L = number of periods in the season

y = observation

S = smoothed observation

b = trend factor

I = seasonal index

F = forecast at *m* periods ahead

t = index denoting time period

and α, β, and γ are constants that must be estimated in such a way that the MSE of the error is minimized.

Mean Absolute Deviation (MAD)

Sum of the absolute values of the individual forecast errors, divided by the number of periods of data (*n*):

$$MAD = \frac{\sum |Actual - Forecast|}{n}$$

Mean Squared Error (MSE)

$$MSE = \frac{\sum (\text{forecast errors})^2}{n}$$

Mean Absolute Percent Error (MAPE)

$$MAPE = \frac{\sum_{i=1}^{n} 100 \left| Actual_t - Forecast_t \right| / Actual_t}{n}$$

Tracking Signal

$$\text{Tracking signal} = \frac{\text{Cumulative error}}{MAD}$$

$$= \frac{\sum(\text{actual demand on period } i - \text{forecast demand in period } i)}{MAD}$$

Production Planning Methods

Takt Time

Takt time = total work time available/(units required/day or year) or available work time/demand rate

This is often also used interchangeably with 1/production rate.

Kanban

Kanban Inventory Cost

$$K(Q) = \frac{AD}{Q} + \frac{hQ}{2} + h(SS)$$

where

K(Q) = total annual cost

Q = batch size

A = setup cost

D = annual demand

h = holding cost

SS = average annual safety stock level

Safety Stock (SS)

$$SS = Z_\alpha \sqrt{\sigma_D^2 L}$$

where

Z_α = number of standard deviations associated with a level of protection α

σ_D^2 = variance of demand

L = deterministic lead time

Number of Kanbans

$$k = \frac{DL + s}{a}$$

where

D = expected demand of parts per unit time

L = deterministic lead time

a = number of units in a standard container

s = safety stock in units

Maximum Inventory Level

$ak = DL + s$

where

D = expected demand of parts per unit time

L = deterministic lead time

a = number of units in a standard container

k = number of containers or kanbans

s = safety stock in units

Machine Capacity

Total Effective Equipment Performance (TEEP) considers maximum time to be All Available Time for the machine.

TEEP = Performance × Quality × Availability

where Availability = Actual Production Time/All Time

Overall Operations Effectiveness (OOE) takes unscheduled time into account, looking at Total Operations Time as the maximum.

OOE = Performance × Quality × Availability

where Availability = Actual Production Time/Operating Time

Overall Equipment Effectiveness (OEE) only considers scheduled time. If a machine is down due to maintenance and it is not scheduled for work, OEE ignores this time.

OEE = Performance × Quality × Availability

where Availability = Actual Production Time/Scheduled Time

Engineering Economics

Compound Interest Tables

Factor Name	Converts	Symbol	Formula
Single Payment Compound Amount	to *F* given *P*	(*F*/*P*, *i*%, *n*)	$(1 + i)^n$
Single Payment Present Worth	to *P* given *F*	(*P*/*F*, *i*%, *n*)	$(1 + i)^{-n}$
Uniform Series Sinking Fund	to *A* given *F*	(*A*/*F*, *i*%, *n*)	$\frac{i}{(1+i)^n - 1}$
Capital Recovery	to *A* given *P*	(*A*/*P*, *i*%, *n*)	$\frac{i(1+i)^n}{(1+i)^n - 1}$
Uniform Series Compound Amount	to *F* given *A*	(*F*/*A*, *i*%, *n*)	$\frac{(1+i)^n - 1}{i}$
Uniform Series Present Worth	to *P* given *A*	(*P*/*A*, *i*%, *n*)	$\frac{(1+i)^n - 1}{i(1+i)^n}$
Uniform Gradient Present Worth	to *P* given *G*	(*P*/*G*, *i*%, *n*)	$\frac{(1+i)^n - 1}{i^2(1+i)^n} - \frac{n}{i(1+i)^n}$
Uniform Gradient † Future Worth	to *F* given *G*	(*F*/*G*, *i*%, *n*)	$\frac{(1+i)^n - 1}{i^2} - \frac{n}{i}$
Uniform Gradient Uniform Series	to *A* given *G*	(*A*/*G*, *i*%, *n*)	$\frac{1}{i} - \frac{n}{(1+i)^n - 1}$

$$\dagger \; F/G = \frac{F/A - n}{i} = (F/A) \times (A/G)$$

Nomenclature and Definitions

A = Uniform amount per interest period
B = Benefit
BV = Book Value
C = Cost
d = Inflation adjusted interest rate per interest period
D_j = Depreciation in year *j*
EV = Expected value
F = Future worth, value, or amount
f = General inflation rate per interest period
G = Uniform gradient amount per interest period
i = Interest rate per interest period
i_e = Annual effective interest rate
MARR = Minimum acceptable/attractive rate of return
m = Number of compounding periods per year
n = Number of compounding periods; or expected life
P = Present worth, value, or amount
r = Nominal annual interest rate
S_n = Expected salvage value in year *n*

Non-Annual Compounding

$$i_e = \left(1 + \frac{r}{m}\right)^m - 1$$

Discount Factors for Continuous Compounding

$(F/P, r\%, n) = e^{rn}$

$(P/F, r\%, n) = e^{-rn}$

$(A/F, r\%, n) = \frac{e^r - 1}{e^{rn} - 1}$

$(F/A, r\%, n) = \frac{e^{rn} - 1}{e^r - 1}$

$(A/P, r\%, n) = \frac{e^r - 1}{1 - e^{-rn}}$

$(P/A, r\%, n) = \frac{1 - e^{-rn}}{e^r - 1}$

Inflation

To account for inflation, the dollars are deflated by the general inflation rate per interest period *f*, and then they are shifted over the time scale using the interest rate per interest period *i*. Use an inflation adjusted interest rate per interest period *d* for computing present worth values *P*.

The formula for *d* is $d = i + f + (i \times f)$

Capitalized Costs

Capitalized Costs = $P = A/i$

Depreciation

Straight Line

$$D_j = \frac{C - S_n}{n}$$

Book Value

BV = initial cost – ΣD_j

Modified Accelerated Cost Recovery System (ACRS)

D_j = (factor) C

Modified ACRS Factors

	Recovery Period (Years)			
	3	5	7	10
Year	Recovery Rate (Percent)			
1	33.3	20.0	14.3	10.0
2	44.5	32.0	24.5	18.0
3	14.8	19.2	17.5	14.4
4	7.4	11.5	12.5	11.5
5		11.5	8.9	9.2
6		5.8	8.9	7.4
7			8.9	6.6
8			4.5	6.6
9				6.5
10				6.5
11				3.3

Factor Table - *i* = 0.50%

N	*P/F*	*P/A*	*P/G*	*F/P*	*F/A*	*A/P*	*A/F*	*A/G*
1	0.9950	0.9950	0.0000	1.0050	1.0000	1.0050	1.0000	0.0000
2	0.9901	1.9851	0.9901	1.0100	2.0050	0.5038	0.4988	0.4988
3	0.9851	2.9702	2.9604	1.0151	3.0150	0.3367	0.3317	0.9967
4	0.9802	3.9505	5.9011	1.0202	4.0301	0.2531	0.2481	1.4938
5	**0.9754**	**4.9259**	**9.8026**	**1.0253**	**5.0503**	**0.2030**	**0.1980**	**1.9900**
6	0.9705	5.8964	14.6552	1.0304	6.0755	0.1696	0.1646	2.4855
7	0.9657	6.8621	20.4493	1.0355	7.1059	0.1457	0.1407	2.9801
8	0.9609	7.8230	27.1755	1.0407	8.1414	0.1278	0.1228	3.4738
9	0.9561	8.7791	34.8244	1.0459	9.1821	0.1139	0.1089	3.9668
10	**0.9513**	**9.7304**	**43.3865**	**1.0511**	**10.2280**	**0.1028**	**0.0978**	**4.4589**
11	0.9466	10.6770	52.8526	1.0564	11.2792	0.0937	0.0887	4.9501
12	0.9419	11.6189	63.2136	1.0617	12.3356	0.0861	0.0811	5.4406
13	0.9372	12.5562	74.4602	1.0670	13.3972	0.0796	0.0746	5.9302
14	0.9326	13.4887	86.5835	1.0723	14.4642	0.0741	0.0691	6.4190
15	**0.9279**	**14.4166**	**99.5743**	**1.0777**	**15.5365**	**0.0694**	**0.0644**	**6.9069**
16	0.9233	15.3399	113.4238	1.0831	16.6142	0.0652	0.0602	7.3940
17	0.9187	16.2586	128.1231	1.0885	17.6973	0.0615	0.0565	7.8803
18	0.9141	17.1728	143.6634	1.0939	18.7858	0.0582	0.0532	8.3658
19	0.9096	18.0824	160.0360	1.0994	19.8797	0.0553	0.0503	8.8504
20	**0.9051**	**18.9874**	**177.2322**	**1.1049**	**20.9791**	**0.0527**	**0.0477**	**9.3342**
21	0.9006	19.8880	195.2434	1.1104	22.0840	0.0503	0.0453	9.8172
22	0.8961	20.7841	214.0611	1.1160	23.1944	0.0481	0.0431	10.2993
23	0.8916	21.6757	233.6768	1.1216	24.3104	0.0461	0.0411	10.7806
24	0.8872	22.5629	254.0820	1.1272	25.4320	0.0443	0.0393	11.2611
25	**0.8828**	**23.4456**	**275.2686**	**1.1328**	**26.5591**	**0.0427**	**0.0377**	**11.7407**
30	0.8610	27.7941	392.6324	1.1614	32.2800	0.0360	0.0310	14.1265
40	0.8191	36.1722	681.3347	1.2208	44.1588	0.0276	0.0226	18.8359
50	0.7793	44.1428	1,035.6966	1.2832	56.6452	0.0227	0.0177	23.4624
60	0.7414	51.7256	1,448.6458	1.3489	69.7700	0.0193	0.0143	28.0064
100	**0.6073**	**78.5426**	**3,562.7934**	**1.6467**	**129.3337**	**0.0127**	**0.0077**	**45.3613**

Factor Table - *i* = 1.00%

N	*P/F*	*P/A*	*P/G*	*F/P*	*F/A*	*A/P*	*A/F*	*A/G*
1	0.9901	0.9901	0.0000	1.0100	1.0000	1.0100	1.0000	0.0000
2	0.9803	1.9704	0.9803	1.0201	2.0100	0.5075	0.4975	0.4975
3	0.9706	2.9410	2.9215	1.0303	3.0301	0.3400	0.3300	0.9934
4	0.9610	3.9020	5.8044	1.0406	4.0604	0.2563	0.2463	1.4876
5	**0.9515**	**4.8534**	**9.6103**	**1.0510**	**5.1010**	**0.2060**	**0.1960**	**1.9801**
6	0.9420	5.7955	14.3205	1.0615	6.1520	0.1725	0.1625	2.4710
7	0.9327	6.7282	19.9168	1.0721	7.2135	0.1486	0.1386	2.9602
8	0.9235	7.6517	26.3812	1.0829	8.2857	0.1307	0.1207	3.4478
9	0.9143	8.5650	33.6959	1.0937	9.3685	0.1167	0.1067	3.9337
10	**0.9053**	**9.4713**	**41.8435**	**1.1046**	**10.4622**	**0.1056**	**0.0956**	**4.4179**
11	0.8963	10.3676	50.8067	1.1157	11.5668	0.0965	0.0865	4.9005
12	0.8874	11.2551	60.5687	1.1268	12.6825	0.0888	0.0788	5.3815
13	0.8787	12.1337	71.1126	1.1381	13.8093	0.0824	0.0724	5.8607
14	0.8700	13.0037	82.4221	1.1495	14.9474	0.0769	0.0669	6.3384
15	**0.8613**	**13.8651**	**94.4810**	**1.1610**	**16.0969**	**0.0721**	**0.0621**	**6.8143**
16	0.8528	14.7179	107.2734	1.1726	17.2579	0.0679	0.0579	7.2886
17	0.8444	15.5623	120.7834	1.1843	18.4304	0.0643	0.0543	7.7613
18	0.8360	16.3983	134.9957	1.1961	19.6147	0.0610	0.0510	8.2323
19	0.8277	17.2260	149.8950	1.2081	20.8109	0.0581	0.0481	8.7017
20	**0.8195**	**18.0456**	**165.4664**	**1.2202**	**22.0190**	**0.0554**	**0.0454**	**9.1694**
21	0.8114	18.8570	181.6950	1.2324	23.2392	0.0530	0.0430	9.6354
22	0.8034	19.6604	198.5663	1.2447	24.4716	0.0509	0.0409	10.0998
23	0.7954	20.4558	216.0660	1.2572	25.7163	0.0489	0.0389	10.5626
24	0.7876	21.2434	234.1800	1.2697	26.9735	0.0471	0.0371	11.0237
25	**0.7798**	**22.0232**	**252.8945**	**1.2824**	**28.2432**	**0.0454**	**0.0354**	**11.4831**
30	0.7419	25.8077	355.0021	1.3478	34.7849	0.0387	0.0277	13.7557
40	0.6717	32.8347	596.8561	1.4889	48.8864	0.0305	0.0205	18.1776
50	0.6080	39.1961	879.4176	1.6446	64.4632	0.0255	0.0155	22.4363
60	0.5504	44.9550	1,192.8061	1.8167	81.6697	0.0222	0.0122	26.5333
100	**0.3697**	**63.0289**	**2,605.7758**	**2.7048**	**170.4814**	**0.0159**	**0.0059**	**41.3426**

Factor Table - *i* = 1.50%

N	P/F	P/A	P/G	F/P	F/A	A/P	A/F	A/G
1	0.9852	0.9852	0.0000	1.0150	1.0000	1.0150	1.0000	0.0000
2	0.9707	1.9559	0.9707	1.0302	2.0150	0.5113	0.4963	0.4963
3	0.9563	2.9122	2.8833	1.0457	3.0452	0.3434	0.3284	0.9901
4	0.9422	3.8544	5.7098	1.0614	4.0909	0.2594	0.2444	1.4814
5	**0.9283**	**4.7826**	**9.4229**	**1.0773**	**5.1523**	**0.2091**	**0.1941**	**1.9702**
6	0.9145	5.6972	13.9956	1.0934	6.2296	0.1755	0.1605	2.4566
7	0.9010	6.5982	19.4018	1.1098	7.3230	0.1516	0.1366	2.9405
8	0.8877	7.4859	26.6157	1.1265	8.4328	0.1336	0.1186	3.4219
9	0.8746	8.3605	32.6125	1.1434	9.5593	0.1196	0.1046	3.9008
10	**0.8617**	**9.2222**	**40.3675**	**1.1605**	**10.7027**	**0.1084**	**0.0934**	**4.3772**
11	0.8489	10.0711	48.8568	1.1779	11.8633	0.0993	0.0843	4.8512
12	0.8364	10.9075	58.0571	1.1956	13.0412	0.0917	0.0767	5.3227
13	0.8240	11.7315	67.9454	1.2136	14.2368	0.0852	0.0702	5.7917
14	0.8118	12.5434	78.4994	1.2318	15.4504	0.0797	0.0647	6.2582
15	**0.7999**	**13.3432**	**89.6974**	**1.2502**	**16.6821**	**0.0749**	**0.0599**	**6.7223**
16	0.7880	14.1313	101.5178	1.2690	17.9324	0.0708	0.0558	7.1839
17	0.7764	14.9076	113.9400	1.2880	19.2014	0.0671	0.0521	7.6431
18	0.7649	15.6726	126.9435	1.3073	20.4894	0.0638	0.0488	8.0997
19	0.7536	16.4262	140.5084	1.3270	21.7967	0.0609	0.0459	8.5539
20	**0.7425**	**17.1686**	**154.6154**	**1.3469**	**23.1237**	**0.0582**	**0.0432**	**9.0057**
21	0.7315	17.9001	169.2453	1.3671	24.4705	0.0559	0.0409	9.4550
22	0.7207	18.6208	184.3798	1.3876	25.8376	0.0537	0.0387	9.9018
23	0.7100	19.3309	200.0006	1.4084	27.2251	0.0517	0.0367	10.3462
24	0.6995	20.0304	216.0901	1.4295	28.6335	0.0499	0.0349	10.7881
25	**0.6892**	**20.7196**	**232.6310**	**1.4509**	**30.0630**	**0.0483**	**0.0333**	**11.2276**
30	0.6398	24.0158	321.5310	1.5631	37.5387	0.0416	0.0266	13.3883
40	0.5513	29.9158	524.3568	1.8140	54.2679	0.0334	0.0184	17.5277
50	0.4750	34.9997	749.9636	2.1052	73.6828	0.0286	0.0136	21.4277
60	0.4093	39.3803	988.1674	2.4432	96.2147	0.0254	0.0104	25.0930
100	**0.2256**	**51.6247**	**1,937.4506**	**4.4320**	**228.8030**	**0.0194**	**0.0044**	**37.5295**

Factor Table - *i* = 2.00%

N	P/F	P/A	P/G	F/P	F/A	A/P	A/F	A/G
1	0.9804	0.9804	0.0000	1.0200	1.0000	1.0200	1.0000	0.0000
2	0.9612	1.9416	0.9612	1.0404	2.0200	0.5150	0.4950	0.4950
3	0.9423	2.8839	2.8458	1.0612	3.0604	0.3468	0.3268	0.9868
4	0.9238	3.8077	5.6173	1.0824	4.1216	0.2626	0.2426	1.4752
5	**0.9057**	**4.7135**	**9.2403**	**1.1041**	**5.2040**	**0.2122**	**0.1922**	**1.9604**
6	0.8880	5.6014	13.6801	1.1262	6.3081	0.1785	0.1585	2.4423
7	0.8706	6.4720	18.9035	1.1487	7.4343	0.1545	0.1345	2.9208
8	0.8535	7.3255	24.8779	1.1717	8.5830	0.1365	0.1165	3.3961
9	0.8368	8.1622	31.5720	1.1951	9.7546	0.1225	0.1025	3.8681
10	**0.8203**	**8.9826**	**38.9551**	**1.2190**	**10.9497**	**0.1113**	**0.0913**	**4.3367**
11	0.8043	9.7868	46.9977	1.2434	12.1687	0.1022	0.0822	4.8021
12	0.7885	10.5753	55.6712	1.2682	13.4121	0.0946	0.0746	5.2642
13	0.7730	11.3484	64.9475	1.2936	14.6803	0.0881	0.0681	5.7231
14	0.7579	12.1062	74.7999	1.3195	15.9739	0.0826	0.0626	6.1786
15	**0.7430**	**12.8493**	**85.2021**	**1.3459**	**17.2934**	**0.0778**	**0.0578**	**6.6309**
16	0.7284	13.5777	96.1288	1.3728	18.6393	0.0737	0.0537	7.0799
17	0.7142	14.2919	107.5554	1.4002	20.0121	0.0700	0.0500	7.5256
18	0.7002	14.9920	119.4581	1.4282	21.4123	0.0667	0.0467	7.9681
19	0.6864	15.6785	131.8139	1.4568	22.8406	0.0638	0.0438	8.4073
20	**0.6730**	**16.3514**	**144.6003**	**1.4859**	**24.2974**	**0.0612**	**0.0412**	**8.8433**
21	0.6598	17.0112	157.7959	1.5157	25.7833	0.0588	0.0388	9.2760
22	0.6468	17.6580	171.3795	1.5460	27.2990	0.0566	0.0366	9.7055
23	0.6342	18.2922	185.3309	1.5769	28.8450	0.0547	0.0347	10.1317
24	0.6217	18.9139	199.6305	1.6084	30.4219	0.0529	0.0329	10.5547
25	**0.6095**	**19.5235**	**214.2592**	**1.6406**	**32.0303**	**0.0512**	**0.0312**	**10.9745**
30	0.5521	22.3965	291.7164	1.8114	40.5681	0.0446	0.0246	13.0251
40	0.4529	27.3555	461.9931	2.2080	60.4020	0.0366	0.0166	16.8885
50	0.3715	31.4236	642.3606	2.6916	84.5794	0.0318	0.0118	20.4420
60	0.3048	34.7609	823.6975	3.2810	114.0515	0.0288	0.0088	23.6961
100	**0.1380**	**43.0984**	**1,464.7527**	**7.2446**	**312.2323**	**0.0232**	**0.0032**	**33.9863**

Factor Table - *i* = 4.00%

n	P/F	P/A	P/G	F/P	F/A	A/P	A/F	A/G
1	0.9615	0.9615	0.0000	1.0400	1.0000	1.0400	1.0000	0.0000
2	0.9246	1.8861	0.9246	1.0816	2.0400	0.5302	0.4902	0.4902
3	0.8890	2.7751	2.7025	1.1249	3.1216	0.3603	0.3203	0.9739
4	0.8548	3.6299	5.2670	1.1699	4.2465	0.2755	0.2355	1.4510
5	**0.8219**	**4.4518**	**8.5547**	**1.2167**	**5.4163**	**0.2246**	**0.1846**	**1.9216**
6	0.7903	5.2421	12.5062	1.2653	6.6330	0.1908	0.1508	2.3857
7	0.7599	6.0021	17.0657	1.3159	7.8983	0.1666	0.1266	2.8433
8	0.7307	6.7327	22.1806	1.3686	9.2142	0.1485	0.1085	3.2944
9	0.7026	7.4353	27.8013	1.4233	10.5828	0.1345	0.0945	3.7391
10	**0.6756**	**8.1109**	**33.8814**	**1.4802**	**12.0061**	**0.1233**	**0.0833**	**4.1773**
11	0.6496	8.7605	40.3772	1.5395	13.4864	0.1141	0.0741	4.6090
12	0.6246	9.3851	47.2477	1.6010	15.0258	0.1066	0.0666	5.0343
13	0.6006	9.9856	54.4546	1.6651	16.6268	0.1001	0.0601	5.4533
14	0.5775	10.5631	61.9618	1.7317	18.2919	0.0947	0.0547	5.8659
15	**0.5553**	**11.1184**	**69.7355**	**1.8009**	**20.0236**	**0.0899**	**0.0499**	**6.2721**
16	0.5339	11.6523	77.7441	1.8730	21.8245	0.0858	0.0458	6.6720
17	0.5134	12.1657	85.9581	1.9479	23.6975	0.0822	0.0422	7.0656
18	0.4936	12.6593	94.3498	2.0258	25.6454	0.0790	0.0390	7.4530
19	0.4746	13.1339	102.8933	2.1068	27.6712	0.0761	0.0361	7.8342
20	**0.4564**	**13.5903**	**111.5647**	**2.1911**	**29.7781**	**0.0736**	**0.0336**	**8.2091**
21	0.4388	14.0292	120.3414	2.2788	31.9692	0.0713	0.0313	8.5779
22	0.4220	14.4511	129.2024	2.3699	34.2480	0.0692	0.0292	8.9407
23	0.4057	14.8568	138.1284	2.4647	36.6179	0.0673	0.0273	9.2973
24	0.3901	15.2470	147.1012	2.5633	39.0826	0.0656	0.0256	9.6479
25	**0.3751**	**15.6221**	**156.1040**	**2.6658**	**41.6459**	**0.0640**	**0.0240**	**9.9925**
30	0.3083	17.2920	201.0618	3.2434	56.0849	0.0578	0.0178	11.6274
40	0.2083	19.7928	286.5303	4.8010	95.0255	0.0505	0.0105	14.4765
50	0.1407	21.4822	361.1638	7.1067	152.6671	0.0466	0.0066	16.8122
60	0.0951	22.6235	422.9966	10.5196	237.9907	0.0442	0.0042	18.6972
100	**0.0198**	**24.5050**	**563.1249**	**50.5049**	**1,237.6237**	**0.0408**	**0.0008**	**22.9800**

Factor Table - *i* = 6.00%

n	P/F	P/A	P/G	F/P	F/A	A/P	A/F	A/G
1	0.9434	0.9434	0.0000	1.0600	1.0000	1.0600	1.0000	0.0000
2	0.8900	1.8334	0.8900	1.1236	2.0600	0.5454	0.4854	0.4854
3	0.8396	2.6730	2.5692	1.1910	3.1836	0.3741	0.3141	0.9612
4	0.7921	3.4651	4.9455	1.2625	4.3746	0.2886	0.2286	1.4272
5	**0.7473**	**4.2124**	**7.9345**	**1.3382**	**5.6371**	**0.2374**	**0.1774**	**1.8836**
6	0.7050	4.9173	11.4594	1.4185	6.9753	0.2034	0.1434	2.3304
7	0.6651	5.5824	15.4497	1.5036	8.3938	0.1791	0.1191	2.7676
8	0.6274	6.2098	19.8416	1.5938	9.8975	0.1610	0.1010	3.1952
9	0.5919	6.8017	24.5768	1.6895	11.4913	0.1470	0.0870	3.6133
10	**0.5584**	**7.3601**	**29.6023**	**1.7908**	**13.1808**	**0.1359**	**0.0759**	**4.0220**
11	0.5268	7.8869	34.8702	1.8983	14.9716	0.1268	0.0668	4.4213
12	0.4970	8.3838	40.3369	2.0122	16.8699	0.1193	0.0593	4.8113
13	0.4688	8.8527	45.9629	2.1329	18.8821	0.1130	0.0530	5.1920
14	0.4423	9.2950	51.7128	2.2609	21.0151	0.1076	0.0476	5.5635
15	**0.4173**	**9.7122**	**57.5546**	**2.3966**	**23.2760**	**0.1030**	**0.0430**	**5.9260**
16	0.3936	10.1059	63.4592	2.5404	25.6725	0.0990	0.0390	6.2794
17	0.3714	10.4773	69.4011	2.6928	28.2129	0.0954	0.0354	6.6240
18	0.3505	10.8276	75.3569	2.8543	30.9057	0.0924	0.0324	6.9597
19	0.3305	11.1581	81.3062	3.0256	33.7600	0.0896	0.0296	7.2867
20	**0.3118**	**11.4699**	**87.2304**	**3.2071**	**36.7856**	**0.0872**	**0.0272**	**7.6051**
21	0.2942	11.7641	93.1136	3.3996	39.9927	0.0850	0.0250	7.9151
22	0.2775	12.0416	98.9412	3.6035	43.3923	0.0830	0.0230	8.2166
23	0.2618	12.3034	104.7007	3.8197	46.9958	0.0813	0.0213	8.5099
24	0.2470	12.5504	110.3812	4.0489	50.8156	0.0797	0.0197	8.7951
25	**0.2330**	**12.7834**	**115.9732**	**4.2919**	**54.8645**	**0.0782**	**0.0182**	**9.0722**
30	0.1741	13.7648	142.3588	5.7435	79.0582	0.0726	0.0126	10.3422
40	0.0972	15.0463	185.9568	10.2857	154.7620	0.0665	0.0065	12.3590
50	0.0543	15.7619	217.4574	18.4202	290.3359	0.0634	0.0034	13.7964
60	0.0303	16.1614	239.0428	32.9877	533.1282	0.0619	0.0019	14.7909
100	**0.0029**	**16.6175**	**272.0471**	**339.3021**	**5,638.3681**	**0.0602**	**0.0002**	**16.3711**

Factor Table - *i* = 8.00%

n	*P/F*	*P/A*	*P/G*	*F/P*	*F/A*	*A/P*	*A/F*	*A/G*
1	0.9259	0.9259	0.0000	1.0800	1.0000	1.0800	1.0000	0.0000
2	0.8573	1.7833	0.8573	1.1664	2.0800	0.5608	0.4808	0.4808
3	0.7938	2.5771	2.4450	1.2597	3.2464	0.3880	0.3080	0.9487
4	0.7350	3.3121	4.6501	1.3605	4.5061	0.3019	0.2219	1.4040
5	**0.6806**	**3.9927**	**7.3724**	**1.4693**	**5.8666**	**0.2505**	**0.1705**	**1.8465**
6	0.6302	4.6229	10.5233	1.5869	7.3359	0.2163	0.1363	2.2763
7	0.5835	5.2064	14.0242	1.7138	8.9228	0.1921	0.1121	2.6937
8	0.5403	5.7466	17.8061	1.8509	10.6366	0.1740	0.0940	3.0985
9	0.5002	6.2469	21.8081	1.9990	12.4876	0.1601	0.0801	3.4910
10	**0.4632**	**6.7101**	**25.9768**	**2.1589**	**14.4866**	**0.1490**	**0.0690**	**3.8713**
11	0.4289	7.1390	30.2657	2.3316	16.6455	0.1401	0.0601	4.2395
12	0.3971	7.5361	34.6339	2.5182	18.9771	0.1327	0.0527	4.5957
13	0.3677	7.9038	39.0463	2.7196	21.4953	0.1265	0.0465	4.9402
14	0.3405	8.2442	43.4723	2.9372	24.2149	0.1213	0.0413	5.2731
15	**0.3152**	**8.5595**	**47.8857**	**3.1722**	**27.1521**	**0.1168**	**0.0368**	**5.5945**
16	0.2919	8.8514	52.2640	3.4259	30.3243	0.1130	0.0330	5.9046
17	0.2703	9.1216	56.5883	3.7000	33.7502	0.1096	0.0296	6.2037
18	0.2502	9.3719	60.8426	3.9960	37.4502	0.1067	0.0267	6.4920
19	0.2317	9.6036	65.0134	4.3157	41.4463	0.1041	0.0241	6.7697
20	**0.2145**	**9.8181**	**69.0898**	**4.6610**	**45.7620**	**0.1019**	**0.0219**	**7.0369**
21	0.1987	10.0168	73.0629	5.0338	50.4229	0.0998	0.0198	7.2940
22	0.1839	10.2007	76.9257	5.4365	55.4568	0.0980	0.0180	7.5412
23	0.1703	10.3711	80.6726	5.8715	60.8933	0.0964	0.0164	7.7786
24	0.1577	10.5288	84.2997	6.3412	66.7648	0.0950	0.0150	8.0066
25	**0.1460**	**10.6748**	**87.8041**	**6.8485**	**73.1059**	**0.0937**	**0.0137**	**8.2254**
30	0.0994	11.2578	103.4558	10.0627	113.2832	0.0888	0.0088	9.1897
40	0.0460	11.9246	126.0422	21.7245	259.0565	0.0839	0.0039	10.5699
50	0.0213	12.2335	139.5928	46.9016	573.7702	0.0817	0.0017	11.4107
60	0.0099	12.3766	147.3000	101.2571	1,253.2133	0.0808	0.0008	11.9015
100	**0.0005**	**12.4943**	**155.6107**	**2,199.7613**	**27,484.5157**	**0.0800**		**12.4545**

Factor Table - *i* = 10.00%

n	*P/F*	*P/A*	*P/G*	*F/P*	*F/A*	*A/P*	*A/F*	*A/G*
1	0.9091	0.9091	0.0000	1.1000	1.0000	1.1000	1.0000	0.0000
2	0.8264	1.7355	0.8264	1.2100	2.1000	0.5762	0.4762	0.4762
3	0.7513	2.4869	2.3291	1.3310	3.3100	0.4021	0.3021	0.9366
4	0.6830	3.1699	4.3781	1.4641	4.6410	0.3155	0.2155	1.3812
5	**0.6209**	**3.7908**	**6.8618**	**1.6105**	**6.1051**	**0.2638**	**0.1638**	**1.8101**
6	0.5645	4.3553	9.6842	1.7716	7.7156	0.2296	0.1296	2.2236
7	0.5132	4.8684	12.7631	1.9487	9.4872	0.2054	0.1054	2.6216
8	0.4665	5.3349	16.0287	2.1436	11.4359	0.1874	0.0874	3.0045
9	0.4241	5.7590	19.4215	2.3579	13.5735	0.1736	0.0736	3.3724
10	**0.3855**	**6.1446**	**22.8913**	**2.5937**	**15.9374**	**0.1627**	**0.0627**	**3.7255**
11	0.3505	6.4951	26.3962	2.8531	18.5312	0.1540	0.0540	4.0641
12	0.3186	6.8137	29.9012	3.1384	21.3843	0.1468	0.0468	4.3884
13	0.2897	7.1034	33.3772	3.4523	24.5227	0.1408	0.0408	4.6988
14	0.2633	7.3667	36.8005	3.7975	27.9750	0.1357	0.0357	4.9955
15	**0.2394**	**7.6061**	**40.1520**	**4.1772**	**31.7725**	**0.1315**	**0.0315**	**5.2789**
16	0.2176	7.8237	43.4164	4.5950	35.9497	0.1278	0.0278	5.5493
17	0.1978	8.0216	46.5819	5.5045	40.5447	0.1247	0.0247	5.8071
18	0.1799	8.2014	49.6395	5.5599	45.5992	0.1219	0.0219	6.0526
19	0.1635	8.3649	52.5827	6.1159	51.1591	0.1195	0.0195	6.2861
20	**0.1486**	**8.5136**	**55.4069**	**6.7275**	**57.2750**	**0.1175**	**0.0175**	**6.5081**
21	0.1351	8.6487	58.1095	7.4002	64.0025	0.1156	0.0156	6.7189
22	0.1228	8.7715	60.6893	8.1403	71.4027	0.1140	0.0140	6.9189
23	0.1117	8.8832	63.1462	8.9543	79.5430	0.1126	0.0126	7.1085
24	0.1015	8.9847	65.4813	9.8497	88.4973	0.1113	0.0113	7.2881
25	**0.0923**	**9.0770**	**67.6964**	**10.8347**	**98.3471**	**0.1102**	**0.0102**	**7.4580**
30	0.0573	9.4269	77.0766	17.4494	164.4940	0.1061	0.0061	8.1762
40	0.0221	9.7791	88.9525	45.2593	442.5926	0.1023	0.0023	9.0962
50	0.0085	9.9148	94.8889	117.3909	1,163.9085	0.1009	0.0009	9.5704
60	0.0033	9.9672	97.7010	304.4816	3,034.8164	0.1003	0.0003	9.8023
100	**0.0001**	**9.9993**	**99.9202**	**13,780.6123**	**137,796.1234**	**0.1000**		**9.9927**

Factor Table - *i* = 12.00%

n	*P/F*	*P/A*	*P/G*	*F/P*	*F/A*	*A/P*	*A/F*	*A/G*
1	0.8929	0.8929	0.0000	1.1200	1.0000	1.1200	1.0000	0.0000
2	0.7972	1.6901	0.7972	1.2544	2.1200	0.5917	0.4717	0.4717
3	0.7118	2.4018	2.2208	1.4049	3.3744	0.4163	0.2963	0.9246
4	0.6355	3.0373	4.1273	1.5735	4.7793	0.3292	0.2092	1.3589
5	**0.5674**	**3.6048**	**6.3970**	**1.7623**	**6.3528**	**0.2774**	**0.1574**	**1.7746**
6	0.5066	4.1114	8.9302	1.9738	8.1152	0.2432	0.1232	2.1720
7	0.4523	4.5638	11.6443	2.2107	10.0890	0.2191	0.0991	2.5515
8	0.4039	4.9676	14.4714	2.4760	12.2997	0.2013	0.0813	2.9131
9	0.3606	5.3282	17.3563	2.7731	14.7757	0.1877	0.0677	3.2574
10	**0.3220**	**5.6502**	**20.2541**	**3.1058**	**17.5487**	**0.1770**	**0.0570**	**3.5847**
11	0.2875	5.9377	23.1288	3.4785	20.6546	0.1684	0.0484	3.8953
12	0.2567	6.1944	25.9523	3.8960	24.1331	0.1614	0.0414	4.1897
13	0.2292	6.4235	28.7024	4.3635	28.0291	0.1557	0.0357	4.4683
14	0.2046	6.6282	31.3624	4.8871	32.3926	0.1509	0.0309	4.7317
15	**0.1827**	**6.8109**	**33.9202**	**5.4736**	**37.2797**	**0.1468**	**0.0268**	**4.9803**
16	0.1631	6.9740	36.3670	6.1304	42.7533	0.1434	0.0234	5.2147
17	0.1456	7.1196	38.6973	6.8660	48.8837	0.1405	0.0205	5.4353
18	0.1300	7.2497	40.9080	7.6900	55.7497	0.1379	0.0179	5.6427
19	0.1161	7.3658	42.9979	8.6128	63.4397	0.1358	0.0158	5.8375
20	**0.1037**	**7.4694**	**44.9676**	**9.6463**	**72.0524**	**0.1339**	**0.0139**	**6.0202**
21	0.0926	7.5620	46.8188	10.8038	81.6987	0.1322	0.0122	6.1913
22	0.0826	7.6446	48.5543	12.1003	92.5026	0.1308	0.0108	6.3514
23	0.0738	7.7184	50.1776	13.5523	104.6029	0.1296	0.0096	6.5010
24	0.0659	7.7843	51.6929	15.1786	118.1552	0.1285	0.0085	6.6406
25	**0.0588**	**7.8431**	**53.1046**	**17.0001**	**133.3339**	**0.1275**	**0.0075**	**6.7708**
30	0.0334	8.0552	58.7821	29.9599	241.3327	0.1241	0.0041	7.2974
40	0.0107	8.2438	65.1159	93.0510	767.0914	0.1213	0.0013	7.8988
50	0.0035	8.3045	67.7624	289.0022	2,400.0182	0.1204	0.0004	8.1597
60	0.0011	8.3240	68.8100	897.5969	7,471.6411	0.1201	0.0001	8.2664
100		**8.3332**	**69.4336**	**83,522.2657**	**696,010.5477**	**0.1200**		**8.3321**

Factor Table - *i* = 18.00%

n	*P/F*	*P/A*	*P/G*	*F/P*	*F/A*	*A/P*	*A/F*	*A/G*
1	0.8475	0.8475	0.0000	1.1800	1.0000	1.1800	1.0000	0.0000
2	0.7182	1.5656	0.7182	1.3924	2.1800	0.6387	0.4587	0.4587
3	0.6086	2.1743	1.9354	1.6430	3.5724	0.4599	0.2799	0.8902
4	0.5158	2.6901	3.4828	1.9388	5.2154	0.3717	0.1917	1.2947
5	**0.4371**	**3.1272**	**5.2312**	**2.2878**	**7.1542**	**0.3198**	**0.1398**	**1.6728**
6	0.3704	3.4976	7.0834	2.6996	9.4423	0.2859	0.1059	2.0252
7	0.3139	3.8115	8.9670	3.1855	12.1415	0.2624	0.0824	2.3526
8	0.2660	4.0776	10.8292	3.7589	15.3270	0.2452	0.0652	2.6558
9	0.2255	4.3030	12.6329	4.4355	19.0859	0.2324	0.0524	2.9358
10	**0.1911**	**4.4941**	**14.3525**	**5.2338**	**23.5213**	**0.2225**	**0.0425**	**3.1936**
11	0.1619	4.6560	15.9716	6.1759	28.7551	0.2148	0.0348	3.4303
12	0.1372	4.7932	17.4811	7.2876	34.9311	0.2086	0.0286	3.6470
13	0.1163	4.9095	18.8765	8.5994	42.2187	0.2037	0.0237	3.8449
14	0.0985	5.0081	20.1576	10.1472	50.8180	0.1997	0.0197	4.0250
15	**0.0835**	**5.0916**	**21.3269**	**11.9737**	**60.9653**	**0.1964**	**0.0164**	**4.1887**
16	0.0708	5.1624	22.3885	14.1290	72.9390	0.1937	0.0137	4.3369
17	0.0600	5.2223	23.3482	16.6722	87.0680	0.1915	0.0115	4.4708
18	0.0508	5.2732	24.2123	19.6731	103.7403	0.1896	0.0096	4.5916
19	0.0431	5.3162	24.9877	23.2144	123.4135	0.1881	0.0081	4.7003
20	**0.0365**	**5.3527**	**25.6813**	**27.3930**	**146.6280**	**0.1868**	**0.0068**	**4.7978**
21	0.0309	5.3837	26.3000	32.3238	174.0210	0.1857	0.0057	4.8851
22	0.0262	5.4099	26.8506	38.1421	206.3448	0.1848	0.0048	4.9632
23	0.0222	5.4321	27.3394	45.0076	244.4868	0.1841	0.0041	5.0329
24	0.0188	5.4509	27.7725	53.1090	289.4944	0.1835	0.0035	5.0950
25	**0.0159**	**5.4669**	**28.1555**	**62.6686**	**342.6035**	**0.1829**	**0.0029**	**5.1502**
30	0.0070	5.5168	29.4864	143.3706	790.9480	0.1813	0.0013	5.3448
40	0.0013	5.5482	30.5269	750.3783	4,163.2130	0.1802	0.0002	5.5022
50	0.0003	5.5541	30.7856	3,927.3569	21,813.0937	0.1800		5.5428
60	0.0001	5.5553	30.8465	20,555.1400	114,189.6665	0.1800		5.5526
100		**5.5556**	**30.8642**	**15,424,131.91**	**85,689,616.17**	**0.1800**		**5.5555**

Production Scheduling Methods

Job Sequencing

$$\text{Average completion time} = \frac{\text{Sum of total flow time}}{\text{Number of jobs}}$$

$$\text{Utilization metric} = \frac{\text{Total job work time}}{\text{Sum of total flow time}}$$

$$\text{Average number of jobs in the system} = \frac{\text{Sum of total flow time}}{\text{Total job work time}}$$

$$\text{Average job lateness} = \frac{\text{Total late days}}{\text{Number of jobs}}$$

Johnson's Rule

1. Select the job with the shortest time, from the list of the jobs, and its time at each work center
2. If the shortest job time is the time at the first work center, schedule it first, otherwise schedule it last. Break ties arbitrarily.
3. Eliminate that job from consideration
4. Repeat 1, 2, and 3 until all jobs have been scheduled.

Inventory Management and Control

Economic Order Quantity

For instantaneous replenishment with constant demand rate, known holding and ordering costs, and an infinite stockout cost

$$EOQ = \sqrt{\frac{2AD}{h}}$$

where

A = cost to place one order

D = number of units used per year

h = holding cost per unit per year

Economic Manufacturing Quantity

Under the same conditions as EOQ with a finite replenishment rate, the economic manufacturing quantity is given by

$$EMQ = \sqrt{\frac{2AD}{h\left(1 - \frac{D}{R}\right)}}$$

where

R = replenishment rate

Mixed Model Relationships

Minimum Operators for Mixed Model Assembly Line

$$n = ceiling\left(\frac{\sum_j\left(N_j\ \sum_i t_{ij}\right)}{T}\right)$$

where

n = minimum number of operators required to produce the given number of units in time period T

j = model being produced

N_j = total units to be produced for model j in time period T

t_{ij} = duration of the i work element in the j model

Economic Part Period

$$Economic\ Part\ Period = \frac{Order\ Cost}{Inventory\ Carrying\ Cost} = \frac{Order\ Cost}{Inventory\ Carrying\ Rate \times Unit\ Cost}$$

Least Period Cost (Silver-Meal)

$$Minimize\ C(j)\ = (K + hr_2 + 2hr_3 + \cdots + \frac{(j-1)hr_j}{r_1 + r_2 + \cdots + r_j})$$

where

$C(j)$ = average holding cost and setup cost per period

K = order cost or setup cost

h = holding cost

r = demand in period

Average Inventory Level Under Safety Stock

$$Average\ Inventory\ Level = \frac{Q}{2} + SS$$

where

Q = order quantity

SS = safety stock

Distribution Methods

Assignment Problem Model

Minimize

$$Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij},$$

Subject to

$$\sum_{j=1}^{n} x_{ij} = 1 \; for \; i = 1,2, \dots , n$$

$$\sum_{i=1}^{n} x_{ij} = 1 \; for \; j = 1,2, \dots , n$$

where

$Z =$ total cost

$$x_{ij} = \begin{cases} 1, & \text{if assignee } i \text{ performs task } j, \\ 0, & \text{if not} \end{cases}$$

c_{ij} = cost of assigning a task j to an assignee i

n = number of assignments and number of tasks

LTL Characteristics

LTL freight shipping reduces costs for shippers by using shared truck payload. Freight shipments are typically:

1. 150 lb or more
2. larger than a parcel but smaller than a truckload
3. palletized or crated, which reduce both shipping rates and damage risks
4. a fraction of the cost of a full truckload shipment
5. sent through local terminals and distribution centers
6. delivered in the mornings, with pickups occurring in the afternoon and at night
7. slower than full truckload transit times because of the shared payload
8. the most popular way to ship

Adapted from www.freightcenter.com/services.ltl-freight

Freight Class	Cost	Notes, Examples
Class 50 – Clean Freight	Lowest Cost	Fits on standard shrink-wrapped 4 × 4 pallet, very durable
Class 55		Bricks, cement, mortar, hardwood flooring
Class 60		Car accessories and car parts
Class 65		Car accessories and car parts, bottled beverages, books in boxes
Class 70		Car accessories and car parts, food items, automobile engines
Class 77.5		Tires, bathroom fixtures
Class 85		Crated machinery, cast iron stoves
Class 92.5		Computers, monitors, refrigerators
Class 100		boat covers, car covers, canvas, wine cases, caskets
Class 110		cabinets, framed artwork, table saw
Class 125		Small household appliances
Class 150		Auto sheet metal parts, bookcases
Class 175		Clothing, couches stuffed furniture
Class 200		Auto sheet metal parts, aircraft parts, aluminum table
Class 250		Bamboo furniture, mattress and box spring, plasma TV
Class 300		Wood cabinets, tables, chairs setup, model boats
Class 400		Deer antlers
Class 500 – Low Density or High Value	Highest Cost	Bags of gold dust, ping pong balls

Freight Management Logistics, "What are Freight Classes?" page, www.fmlfreight.com/freight-101/freight-classes.

Truck Load Freight

Based on federal requirements, full truck load freight occurs when one of the following occurs:

- Weight per truck load approaches 40,000 to 45,000 lb, or
- Volume per truck load approaches 4,000 to 4,500 ft^3, or
- More than 26 single height or 52 double stacked pallets per load (using standard pallets of 48 in. by 40 in., or the next most common size, 48 in. by 48 in.).

The costs and benefits should be evaluated the closer the LTL approaches Full Truck Load limits. Intrastate full load freight has unique requirements by state that maybe higher than federal standards. Typically, full truckload is faster and less expensive per pound than LTL freight.

4 Work Design

Line Balancing

$$N_{\min} = \left(\frac{OR \times \sum_i t_i}{OT}\right)$$

$$Idle\,\frac{Time}{Station} = CT - ST$$

$$Percent\ Idle\ Time = \frac{Idle\,\frac{Time}{Cycle}}{N_{actual} \times CT} \times 100$$

$$Line\ Efficiency = \frac{\sum_i t_i}{N \times CT} \times 100$$

where

CT = cycle time = average time between completion of parts

OT = operating time/period

OR = output rate/period

ST = station time

t_i = individual task times

N = number of stations

Work Measurement Systems Techniques

Standard Time Determination

$NT = OT \times R$

$ST = NT \times AF$

$AF = 1 + A_{\text{job}}$

$$AF = \frac{1}{\left(1 - A_{\text{day}}\right)}$$

where

OT = observed time

R = rating

NT = normal time

AF = allowance factor

A_{job} = allowance fraction (percentage) based on *job time*.

A_{day} = allowance fraction (percentage) based on *workday*.

Performance Rating and Variable Fatigue Allowances

$$F = (T - t) \times \frac{100}{T}$$

where

F = coefficient of fatigue

T = time required to perform operation at end of continuous work

t = time required to perform operation at beginning of continuous work

Learning Curve

The time to do the repetition N of a task is given by

$T_N = KN^S$

The average time per unit is given by

$$T_{\text{avg}} = \frac{K}{N(1+s)}\left[(N+0.5)^{(1+s)} - 0.5^{(1+s)}\right]$$

where

K = constant

$s = \frac{\ln(learning\ rate)}{\ln 2}$

Work Sampling

Absolute Error:

$$D = Z_{\propto/2}\sqrt{\frac{p(1-p)}{n}}$$

Relative Error:

$$R = Z_{\propto/2}\sqrt{\frac{1-p}{pn}}$$

where

p = proportion of observed time in an activity

n = sample size

Safety Codes, Standards, and Voluntary Guidelines

Noise Dose and Time-Weighted Average

Noise Dose

$$D = 100 \cdot \left(\frac{C_1}{T_1} + \frac{C_2}{T_2} + \cdots + \frac{C_n}{T_n}\right)$$

where

C = actual time exposed at a noise level

T = permitted time at the specified level

Time Weighted Average (Equivalent 8-Hr Exposure) Sound Level:

$$TWA = 16.61 \times \log_{10}\left(\frac{D}{100}\right) + 90$$

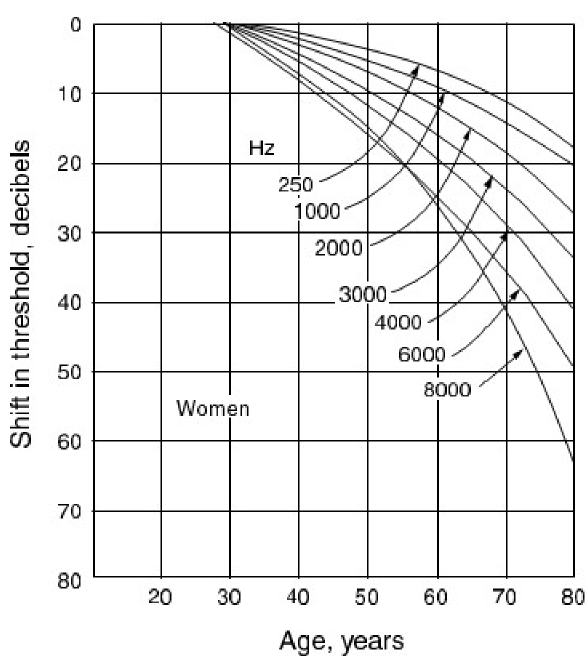
Permissible Noise Exposures

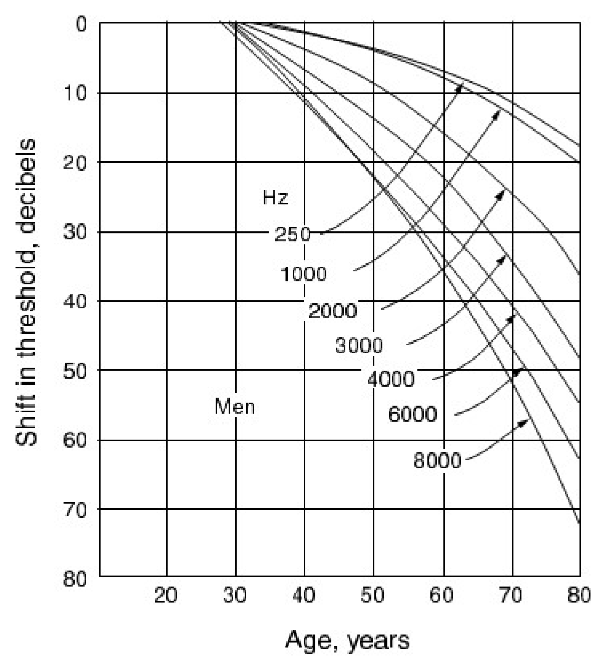
Permissible Noise Exposures (OSHA Occupational Noise Exposure 1910.95)

Duration per day (hours)	Sound level (dBA)
8	90
6	92
4	95
3	97
2	100
1 1/2	102
1	105
1/2	110
1/4 or less	115

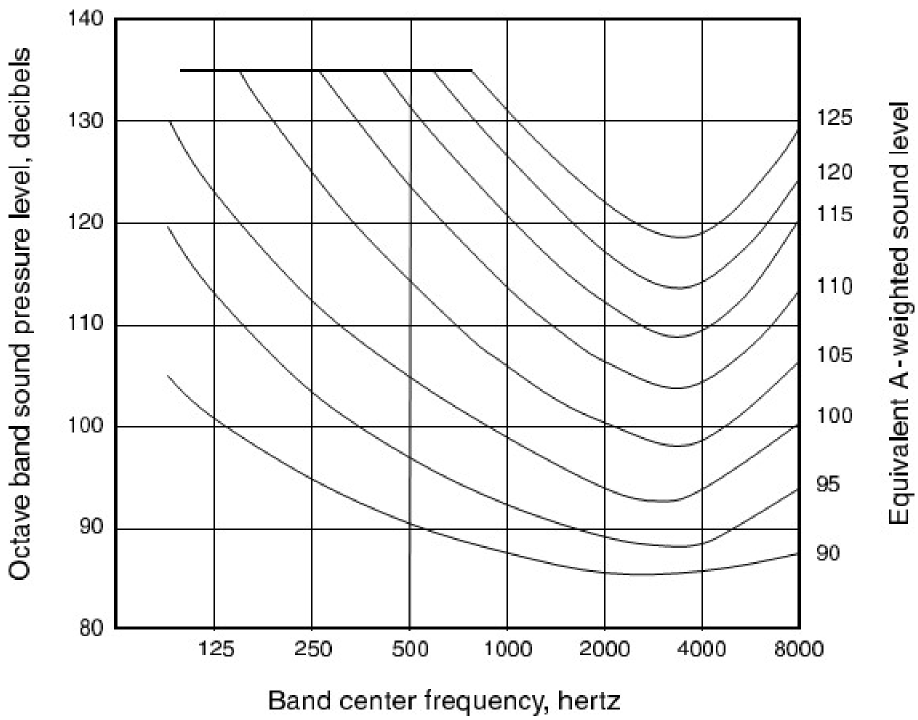
*Exposure to impulsive or impact noise should not exceed 140 dB peak sound pressure level.

Hearing





Normal Hearing Threshold



Equivalent Sound Level Contours (OSHA Occupational Noise Exposure 1910.95)

Noise Pollution

$$SPL\ (dB) = 10 \log_{10} \left(\frac{P^2}{P_0^2} \right)$$

$$SPL_{\text{total}} = 10 \log_{10} \sum 10^{\frac{SPL}{10}}$$

Point Source Attenuation

$$\Delta\ SPL\ (dB) = 10\ \log_{10} \left(\frac{r_1}{r_2} \right)^2$$

Line Source Attenuation

$$\Delta\ SPL\ (dB) = 10\ \log_{10} \left(\frac{r_1}{r_2} \right)$$

where

$SPL\ (dB)$ = sound pressure level, measured in decibels

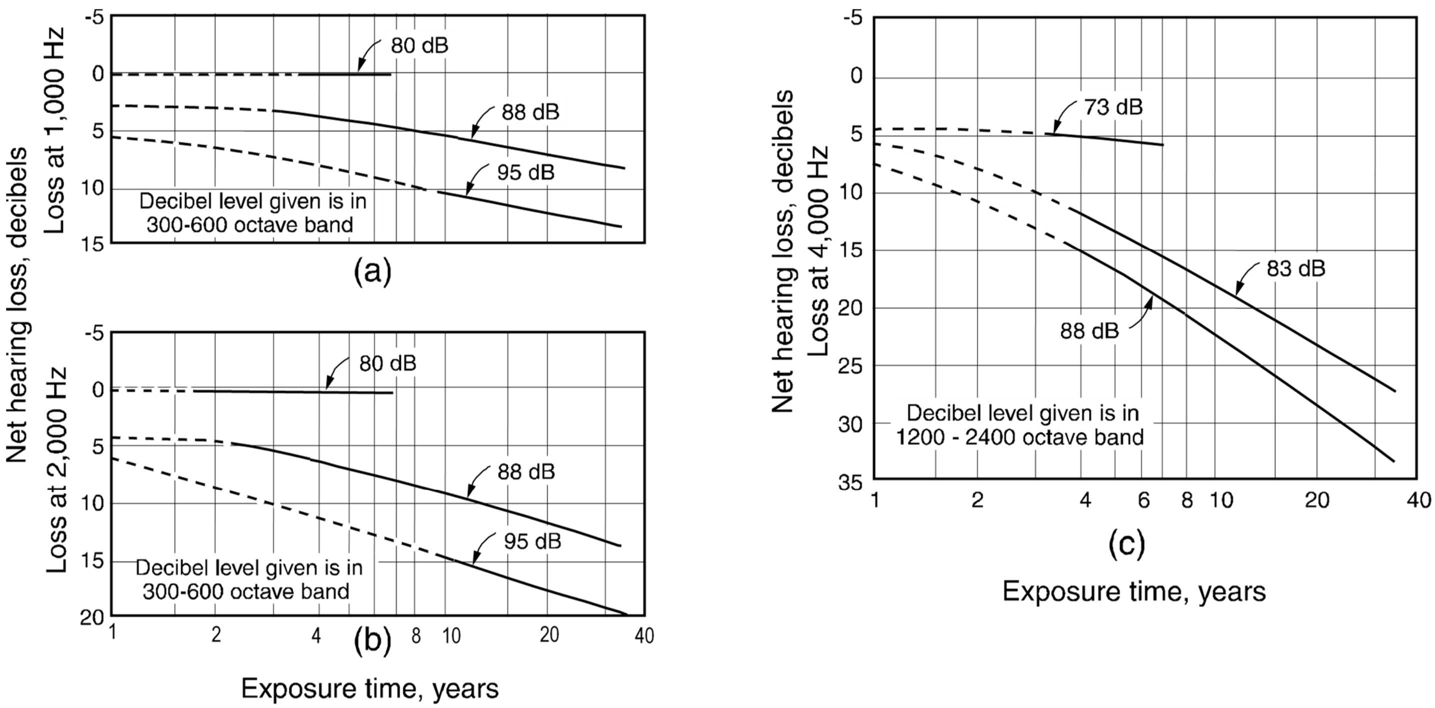
P = sound pressure (Pa)

P_0 = reference sound pressure (2×10^{-5} Pa)

r = distance from source to receptor at a point

Net Hearing Loss

Estimated average trend curves for net hearing loss at 1,000, 2,000, and 4,000 Hz after continuous exposure to steady noise. Data are corrected for age, but not for temporary threshold shift.



Net Hearing Loss

"The Relations of Hearing Loss to Noise Exposure," Exploratory Subcommittee Z24-X-2 of the American Standards Association Z24 Special Committee on Acoustics, Vibration, and Mechanical Shock, sponsored by the Acoustical Study of America, American Standards Association, 1954, pp. 31-33.

Friction

Coefficient of Friction

$$Coefficient\ of\ Friction = \mu = \frac{Friction\ Force}{Normal\ Force}$$

Laws of Friction

$F \leq \mu N$

$F \leq \mu N \cos \alpha$

where

F = force of friction that each surface exerts on each other
μ = coefficient of friction
N = normal force
α = angle of the floor

In general

$F < \mu_s N$, no slip occurring

$F = \mu_s N$, the point of impending slip

$F < \mu_k N$, when slip is occurring

where

μ_s = tan θ = coefficient of static friction

μ_k = coefficient of kinetic friction

Kinetic Friction

$F_k = \mu_k R$

where

F_k = kinetic friction

μ_k = coefficient of kinetic friction

R = force of reaction in the moving body on the surface

Static Friction

$F_s = \mu_s R$

where

F_s = static friction

μ_s = tan θ = coefficient of static friction

θ = angle at which the object begins to slide

R = force of reaction in the moving body on the surface

Limits of Human Capacity

Biomechanics of the Human Body

Basic Equations

$H_x + F_x = 0$

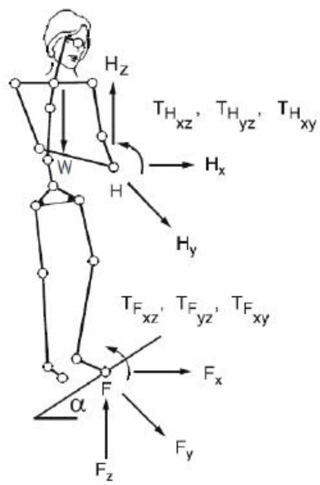
$H_y + F_y = 0$

$H_z + W + F_z = 0$

$T_{H_{xz}} + T_{W_{xz}} + T_{F_{xz}} = 0$

$T_{H_{yz}} + T_{W_{yz}} + T_{F_{yz}} = 0$

$T_{H_{xy}} + T_{F_{xy}} = 0$



Biomechanics of the Human Body

NIOSH Lifting Equation

Recommended Weight Limit (RWL)

RWL = LC x HM x VM x DM x AM x FM x CM

where items are defined in the table.

RWL Basic Constants and Multipliers

		Metric	U.S. Customary
Load Constant	LC	23 kg	51 lb
Horizontal Multiplier	HM	(25/H)	(10/H)
Vertical Multiplier	VM	1-(0.003|V-75|)	1-(0.0075|V-30|)
Distance Multiplier	DM	0.82 + 4.5/D	0.82 + 1.8/D
Asymmetric Multiplier	AM	1-(0.0032A)	1-(0.0032A)
Frequency Multiplier	FM	See table below	See table below
Coupling Multiplier	CM	See table below	See table below

Frequency Multiplier Table

<table>
<tr><td></td><td colspan="2">≤ 8 hr / day</td><td colspan="2">≤ 2 hr / day</td><td colspan="2">≤ 1 hr / day</td></tr>
<tr><td>F, min^{-1}</td><td>V < 30 in.</td><td>V ≥ 30 in.</td><td>V < 30 in.</td><td>V ≥ 30 in.</td><td>V < 30 in.</td><td>V ≥ 30 in.</td></tr>
<tr><td>0.2</td><td colspan="2">0.85</td><td colspan="2">0.95</td><td colspan="2">1.00</td></tr>
<tr><td>0.5</td><td colspan="2">0.81</td><td colspan="2">0.92</td><td colspan="2">0.97</td></tr>
<tr><td>1</td><td colspan="2">0.75</td><td colspan="2">0.88</td><td colspan="2">0.94</td></tr>
<tr><td>2</td><td colspan="2">0.65</td><td colspan="2">0.84</td><td colspan="2">0.91</td></tr>
<tr><td>3</td><td colspan="2">0.55</td><td colspan="2">0.79</td><td colspan="2">0.88</td></tr>
<tr><td>4</td><td colspan="2">0.45</td><td colspan="2">0.72</td><td colspan="2">0.84</td></tr>
<tr><td>5</td><td colspan="2">0.35</td><td colspan="2">0.60</td><td colspan="2">0.80</td></tr>
<tr><td>6</td><td colspan="2">0.27</td><td colspan="2">0.50</td><td colspan="2">0.75</td></tr>
<tr><td>7</td><td colspan="2">0.22</td><td colspan="2">0.42</td><td colspan="2">0.70</td></tr>
<tr><td>8</td><td colspan="2">0.18</td><td colspan="2">0.35</td><td colspan="2">0.60</td></tr>
<tr><td>9</td><td rowspan="2">0.00</td><td>0.15</td><td colspan="2">0.30</td><td colspan="2">0.52</td></tr>
<tr><td>10</td><td>0.13</td><td colspan="2">0.26</td><td colspan="2">0.45</td></tr>
<tr><td>11</td><td colspan="2" rowspan="5">0.00</td><td rowspan="2">0.00</td><td>0.23</td><td colspan="2">0.41</td></tr>
<tr><td>12</td><td>0.21</td><td colspan="2">0.37</td></tr>
<tr><td>13</td><td colspan="2" rowspan="3">0.00</td><td rowspan="3">0.00</td><td>0.34</td></tr>
<tr><td>14</td><td>0.31</td></tr>
<tr><td>15</td><td>0.28</td></tr>
</table>

Coupling Multiplier Table

Coupling Type	V < 30 inches (75 cm)	V ≥ 30 inches (75 cm)
Good	1.00	1.00
Fair	0.95	1.00
Poor	0.90	0.90

"Applications Manual for Revised NIOSH Lifting Equation", January 1, 1994

Lifting Index

$$LI = \frac{LW}{RWL}$$

where

LI = lifting index

LW = load weight

RWL = recommended weight limit

Fitts' Law

$MT = a + bI$

$I = log_2 \frac{D}{W/2}$

where

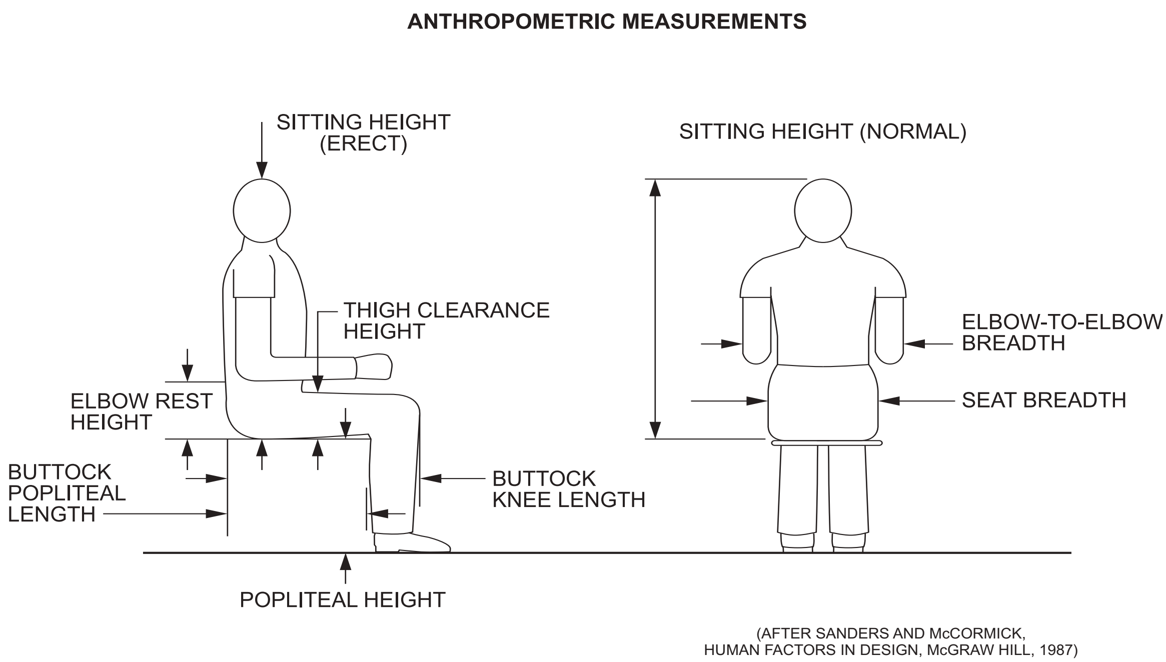
MT = movement time

a and b are constants that are device dependent

D = distance of move

W = width of target

Anthropometric Measurements



Anthropometric Measurements

Sanders and McCormick, *Human Factors in Engineering and Design*, McGraw-Hill, 1987.

US Civilian Body Dimensions by Percentile

U.S. Civilian Body Dimensions, Female/Male, for Ages 20 to 60 Years (Centimeters)				
(See Anthropometric Measurements Figure)	Percentiles			
	5th	50th	95th	Std. Dev.
HEIGHTS				
Stature (height)	149.5 / 161.8	160.5 / 173.6	171.3 / 184.4	6.6 / 6.9
Eye height	138.3 / 151.1	148.9 / 162.4	159.3 / 172.7	6.4 / 6.6
Shoulder (acromion) height	121.1 / 132.3	131.1 / 142.8	141.9 / 152.4	6.1 / 6.1
Elbow height	93.6 / 100.0	101.2 / 109.9	108.8 / 119.0	4.6 / 5.8
Knuckle height	**64.3 / 69.8**	**70.2 / 75.4**	**75.9 / 80.4**	**3.5 / 3.2**
Height, sitting (erect)	78.6 / 84.2	85.0 / 90.6	90.7 / 96.7	3.5 / 3.7
Eye height, sitting	67.5 / 72.6	73.3 / 78.6	78.5 / 84.4	3.3 / 3.6
Shoulder height, sitting	49.2 / 52.7	55.7 / 59.4	61.7 / 65.8	3.8 / 4.0
Elbow rest height, sitting	18.1 / 19.0	23.3 / 24.3	28.1 / 29.4	2.9 / 3.0
Knee height, sitting	**45.2 / 49.3**	**49.8 / 54.3**	**54.5 / 59.3**	**2.7 / 2.9**
Popliteal height, sitting	35.5 / 39.2	39.8 / 44.2	44.3 / 48.8	2.6 / 2.8
Thigh clearance height	10.6 / 11.4	13.7 / 14.4	17.5 / 17.7	1.8 / 1.7
DEPTHS				
Chest depth	21.4 / 21.4	24.2 / 24.2	29.7 / 27.6	2.5 / 1.9
Elbow-fingertip distance	38.5 / 44.1	42.1 / 47.9	46.0 / 51.4	2.2 / 2.2
Buttock-knee length, sitting	51.8 / 54.0	56.9 / 59.4	62.5 / 64.2	3.1 / 3.0
Buttock-popliteal length, sitting	43.0 / 44.2	48.1 / 49.5	53.5 / 54.8	3.1 / 3.0
Forward reach, functional	64.0 / 76.3	71.0 / 82.5	79.0 / 88.3	4.5 / 5.0
BREADTHS				
Elbow-to-elbow breadth	31.5 / 35.0	38.4 / 41.7	49.1 / 50.6	5.4 / 4.6
Seat (hip) breadth, sitting	31.2 / 30.8	36.4 / 35.4	43.7 / 40.6	3.7 / 2.8
HEAD DIMENSIONS				
Head breadth	13.6 / 14.4	14.54 / 15.42	15.5 / 16.4	0.57 / 0.59
Head circumference	52.3 / 53.8	54.9 / 56.8	57.7 / 59.3	1.63 / 1.68
Interpupillary distance	5.1 / 5.5	5.83 / 6.20	6.5 / 6.8	0.4 / 0.39
HAND DIMENSIONS				
Hand length	16.4 / 17.6	17.95 / 19.05	19.8 / 20.6	1.04 / 0.93
Breadth, metacarpal	7.0 / 8.2	7.66 / 8.88	8.4 / 9.8	0.41 / 0.47
Circumference, metacarpal	16.9 / 19.9	18.36 / 21.55	19.9 / 23.5	0.89 / 1.09
Thickness, metacarpal III	2.5 / 2.4	2.77 / 2.76	3.1 / 3.1	0.18 / 0.21
Digit 1				
Breadth, interphalangeal	1.7 / 2.1	1.98 / 2.29	2.1 / 2.5	0.12 / 0.13
Crotch-tip length	4.7 / 5.1	5.36 / 5.88	6.1 / 6.6	0.44 / 0.45
Digit 2				
Breadth, distal joint	1.4 / 1.7	1.55 / 1.85	1.7 / 2.0	0.10 / 0.12
Crotch-tip length	6.1 / 6.8	6.88 / 7.52	7.8 / 8.2	0.52 / 0.46
Digit 3				
Breadth, distal joint	1.4 / 1.7	1.53 / 1.85	1.7 / 2.0	0.09 / 0.12
Crotch-tip length	7.0 / 7.8	7.77 / 8.53	8.7 / 9.5	0.51 / 0.51
Digit 4				
Breadth, distal joint	1.3 / 1.6	1.42 / 1.70	1.6 / 1.9	0.09 / 0.11
Crotch-tip length	6.5 / 7.4	7.29 / 7.99	8.2 / 8.9	0.53 / 0.47
Digit 5				
Breadth, distal joint	1.2 / 1.4	1.32 / 1.57	1.5 / 1.8	0.09/0.12
Crotch-tip length	4.8 / 5.4	5.44 / 6.08	6.2 / 6.99	0.44/0.47
FOOT DIMENSIONS				
Foot length	22.3 / 24.8	24.1 / 26.9	26.2 / 29.0	1.19 / 1.28
Foot breadth	8.1 / 9.0	8.84 / 9.79	9.7 / 10.7	0.50 / 0.53
Lateral malleolus height	5.8 / 6.2	6.78 / 7.03	7.8 / 8.0	0.59 / 0.54
Weight (kg)	46.2 / 56.2	61.1 / 74.0	89.9 / 97.1	13.8 / 12.6

Kroemer, Karl, H. E., "Engineering Anthropometry", *Ergonomics*, Vol. 32, No. 7, pp. 779–780, 1989.

Days Away, Restricted, and Transferred (DART) Rate Calculations

Injury/Illness Incident Rate

$$IR = \frac{N \times 200{,}000}{T}$$

where

IR = total injury/illness incidence rate

N = number of injuries and illnesses

T = total hours worked by all employees during the period in question

Incidence Variable Values

The formula for determining the total injury/illness incident rate is as follows:

IR = (N x 200,000) ÷ *T*

where

IR = total injury/illness incidence rate

N = number of injuries and illnesses

T = total hours worked by all employees during the period in question

SR = total number lost work days/total number of recordable incidents

where

SR = severity rate

5 Quality Engineering

Statistical Process Control

General Definitions

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

$$\bar{\bar{x}} = \frac{\overline{x_1} + \overline{x_2} + \cdots + \overline{x_k}}{k}$$

$$\bar{R} = \frac{R_1 + R_2 + \cdots + R_k}{k}$$

where

x_i = an individual observation

n = the sample size of a group

k = the number of groups

R_i= an individual range

R = (range) the difference between the largest and smallest observations in a sample of size *n*

R Chart

$CL_R = \bar{R}$

$UCL_R = D_4\bar{R}$

$LCL_R = D_3\bar{R}$

$\overline{\mathbf{X}}$ Chart

$CL_X = \bar{\bar{x}}$

$UCL_X = \bar{\bar{x}} + A_2\bar{R}$

$LCL_X = \bar{\bar{x}} + A_2\bar{R}$

Standard Deviation Charts

$UCL_x = \bar{\bar{x}} + A_3\bar{s}$

$CL_x = \bar{\bar{x}}$

$LCL_x = \bar{\bar{x}} - A_3\bar{s}$

$UCL_S = B_4\bar{s}$

$CL_S = \bar{S}$

$LCL_S = B_3\bar{s}$

Approximations

$$\hat{\sigma} = \frac{\bar{R}}{d_2}$$

$$\hat{\sigma} = \frac{\bar{s}}{c_4}$$

$$\sigma_R = d_3\hat{\sigma}$$

$$\sigma_s = \hat{\sigma}\sqrt{1 - c_4^2}$$

where

$\hat{\sigma}$ = an estimate of σ

σ_R = an estimate of the standard deviation of the ranges of the samples

σ_S = an estimate of the standard deviation of the standard deviations of the samples

Factors for Constructing Variables Control Charts

Factors for Constructing Variables Control Charts

n	A	A_2	A_3	c_4	B_3	B_4	B_5	B_6	d_2	d_3	D_1	D_2	D_3	D_4
2	2.121	1.880	2.659	0.798	0.000	3.267	0.000	2.606	1.128	0.853	0.000	3.686	0.000	3.267
3	1.732	1.023	1.954	0.886	0.000	2.568	0.000	2.276	1.693	0.888	0.000	4.358	0.000	2.575
4	1.500	0.729	1.628	0.921	0.000	2.266	0.000	2.088	2.059	0.880	0.000	4.698	0.000	2.282
5	1.342	0.577	1.427	0.940	0.000	2.089	0.000	1.964	2.326	0.864	0.000	4.918	0.000	2.115
6	1.225	0.483	1.287	0.952	0.030	1.970	0.029	1.874	2.534	0.848	0.000	5.078	0.000	2.004
7	1.134	0.419	1.182	0.959	0.118	1.882	0.113	1.806	2.704	0.833	0.204	5.204	0.076	1.924
8	1.061	0.373	1.099	0.965	0.185	1.815	0.179	1.751	2.847	0.820	0.388	5.306	0.136	1.864
9	1.000	0.337	1.032	0.969	0.239	1.761	0.232	1.707	2.970	0.808	0.547	5.393	0.184	1.816
10	0.949	0.308	0.975	0.973	0.284	1.716	0.276	1.669	3.078	0.797	0.687	5.469	0.223	1.777
11	0.905	0.285	0.927	0.975	0.321	1.679	0.313	1.637	3.173	0.787	0.811	5.535	0.256	1.744
12	0.866	0.266	0.886	0.978	0.354	1.646	0.346	1.610	3.258	0.778	0.922	5.594	0.283	1.717
13	0.832	0.249	0.850	0.979	0.382	1.618	0.374	1.585	3.336	0.770	1.025	5.647	0.307	1.693
14	0.802	0.235	0.817	0.981	0.406	1.594	0.399	1.563	3.407	0.763	1.118	5.696	0.328	1.672
15	0.775	0.223	0.789	0.982	0.428	1.572	0.421	1.544	3.472	0.756	1.203	5.741	0.347	1.653
16	0.750	0.212	0.763	0.984	0.448	1.552	0.440	1.526	3.532	0.750	1.282	5.782	0.363	1.637
17	0.728	0.203	0.739	0.985	0.466	1.534	0.458	1.511	3.588	0.744	1.356	5.820	0.378	1.622
18	0.707	0.194	0.718	0.985	0.482	1.518	0.475	1.496	3.640	0.739	1.424	5.856	0.391	1.608
19	0.688	0.187	0.698	0.986	0.497	1.503	0.490	1.483	3.689	0.734	1.487	5.891	0.403	1.597
20	0.671	0.180	0.680	0.987	0.510	1.490	0.504	1.470	3.735	0.729	1.549	5.921	0.415	1.585
21	0.655	0.173	0.663	0.988	0.523	1.477	0.516	1.459	3.778	0.724	1.605	5.951	0.425	1.575
22	0.640	0.167	0.647	0.988	0.534	1.466	0.528	1.448	3.819	0.720	1.659	5.979	0.434	1.566
23	0.626	0.162	0.633	0.9887	0.545	1.455	0.539	1.438	3.858	0.716	1.71	6.006	0.443	1.557
24	0.612	0.157	0.619	0.9892	0.555	1.445	0.549	1.429	3.895	0.712	1.759	6.031	0.451	1.548
25	0.6	0.153	0.606	0.9896	0.565	1.435	0.559	1.42	3.931	0.708	1.806	6.056	0.459	1.541

Duncan, A. J., *Quality Control and Industrial Statistics*, 1974.

For $n > 25$

$A = \frac{3}{\sqrt{n}}$ $\quad A_3 = \frac{3}{c_4\sqrt{n}}$ $\quad c_4 \cong \frac{4(n-1)}{4n-3}$

$B_3 = 1 - \frac{3}{c_4\sqrt{2(n-1)}}$ $\quad B_4 = 1 + \frac{3}{c_4\sqrt{2(n-1)}}$

$B_5 = c_4 - \frac{3}{\sqrt{2(n-1)}}$ $\quad B_6 = c_4 + \frac{3}{\sqrt{2(n-1)}}$

Tests for Out of Control

1. A single point falls outside the (three sigma) control limits.
2. Two out of three successive points fall on the same side of and more than two sigma units from the center line.
3. Four out of five successive points fall on the same side of and more than one sigma unit from the center line.
4. Eight successive points fall on the same side of the center line.

p Chart

$$\hat{p} = \frac{D_i}{n}$$

where

D_i = number of nonconforming observations in the current subgroup *i*

n = number of observations or samples in each subgroup

$$\bar{p} = \frac{\sum_{i=1}^{m} \hat{p}_i}{m}$$

$$UCL_p = \bar{p} + 3\sqrt{\frac{\bar{p}\,(1-\bar{p})}{n}}$$

$$CL_p = \bar{p}$$

$$LCL_p = maximum\left(0, \bar{p} - 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}}\right)$$

where

m = number of subgroups collected

c Chart

$$\bar{c} = \frac{\sum_{i=1}^{m} c_i}{m}$$

$$UCL_c = \bar{c} + 3\sqrt{\bar{c}}$$

$$CL_c = \bar{c}$$

$$LCL_c = maximum\,(0, \bar{c} - 3\sqrt{\bar{c}})$$

where

c_i = number of defects per sample

m = number of samples collected

u Chart

$$\bar{u} = \frac{\sum c}{k}$$
$$UCL = \bar{u} + \left(3\sqrt{\bar{u}}\right)/\sqrt{n}$$
$$CL = \bar{u} = \frac{\sum c}{k}$$
$$LCL = \bar{u} - (3\sqrt{\bar{u}})/\sqrt{n}$$

where

c = number of incidences per subgroup

n = number of "standard areas of opportunity" in a subgroup (*n* may vary)

u = incidences per standard area of opportunity = $\frac{c}{n}$

k = number of subgroups

np Chart

$$np = subgroup\ defective\ count$$

$$n\bar{p} = \frac{sum\ of\ subgroups\ defective\ counts}{number\ of\ subgroups}$$

$$LCL = n\bar{p} - 3\sqrt{n\bar{p}(1-\overline{p})}$$

$$UCL = n\bar{p} + 3\sqrt{n\bar{p}(1-\overline{p})}$$

Capability Ratios (Cp, Cpk)

$$Potential\ Capability = Cp = \frac{USL - LSL}{6\hat{\sigma}}$$

$$Actual\ Capability = Cpk = minimum\left(\frac{USL - \hat{\mu}}{3\hat{\sigma}}, \frac{\hat{\mu} - LSL}{3\hat{\sigma}}\right)$$

Acceptance Sampling

Attribute Sampling

The following provides the sample size with zero defects required to make a confidence statement about the probability of an item being in-spec.

$$n = \frac{ln(1 - Confidence)}{ln(Reliabilty)}$$

Single Sampling

Single sampling plan is the sampling inspection plan in which the lot disposition is based on the inspection of a single sample of size *n*.

Operating Characteristic (OC) Curve for Single Sampling Plan

Type A OC Curve:

$$f(x) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$$

Subject to:
$x = max\{0, n + K - N\}\ to\ min\{K, n\}$

where

X = number of successful objects in a randomly selected sample of size *n* without replacement.

N = number of objects

K = number of objects deemed successful

$N - K$ = number of objects deemed a failure

n = sample size

Type B OC Curve:

Suppose the lot size *N* is large (say infinity). Under this condition, the distribution of the number of defectives *d* in a random sample of *n* items is binomial with parameters *n* and *p*, where *p* is the fraction defective items in the lot. The probability of observing exactly *d* defectives is

$$p(d) = \frac{n!}{d!\,(n-d)!} p^d (1-p)^{n-d}$$

The probability of acceptance

$$P_a\{d \le c\} = \sum_{d=0}^{c} \frac{n!}{d!\,(n-d)!} p^d (1-p)^{n-d}$$

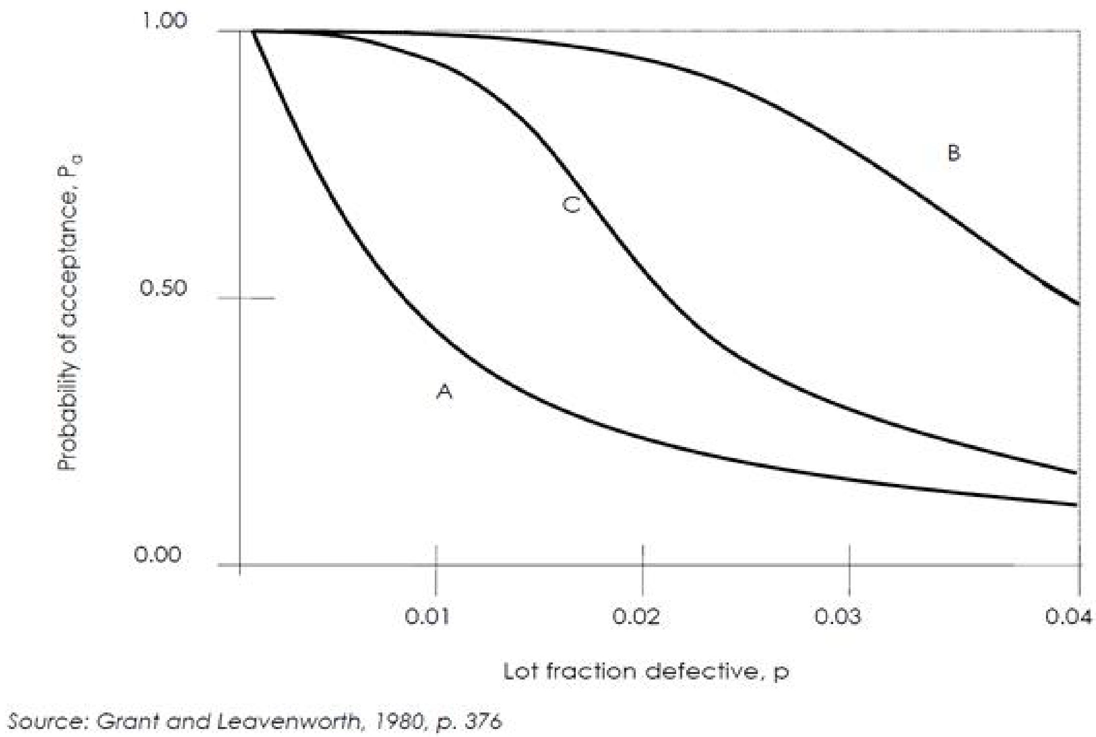
Single Sampling Plan with a Specified Type B OC Curve

$$1 - \alpha = \sum_{d=0}^{c} \frac{n!}{d!\,(n-d)!} p_1^d (1-p_1)^{n-d}$$

$$\beta = \sum_{d=0}^{c} \frac{n!}{d!\,(n-d)!} p_2^d\,(1-p_2)^{n-d}$$

Double Sampling

Double Sampling Operating Characteristic (OC) Curve



Operating Characteristic (OC) Curve for a Double Sampling Curve

Probability of acceptance on the first sample (Curve A)

$$P_a^A = P_a^I = P\{x_1 \leq c_1\} = \sum_{x_1=0}^{c_1} \frac{n_1!}{x_1!\,(n_1 - x_1)!} p^{x_1}(1-p)^{n_1-x_1}$$

Probability of not rejecting the lot on the first sample (Curve B)

$$P_a^B = P_a^I = P\{x_1 \leq c_2\} = \sum_{x_1=0}^{c_2} \frac{n_1!}{x_1!\,(n_1 - x_1)!} p^{x_1}(1-p)^{n_1-x_1}$$

Probability of acceptance for the double sampling plan (Curve C)

$$P_a = P_a^I + P_a^I = P\{x_1 \leq c_1\} + \sum_{i=c_1+1}^{c_2} P\{x_1 = i\}\, P\{x_2 \leq c_2 - i\}$$

The probability of acceptance on the first sample (P_a^I) includes probability for $x_1 \leq c_1$ and the probability of acceptance on the second sample (P_a^I) includes probability of different ways in which the second sample can be obtained.

Average Sample Number (ASN) Curve

$ASN = n_1 + n_2(1 - P_I)$

where

$P_I = P\{lot\ is\ accepted\ on\ the\ first\ sample\} + P\{lot\ is\ rejected\ on\ the\ first\ sample\}$

AOQ (Average Outgoing Quality) Curve

$$AOQ = \frac{[P_a^I(N - n_1) + P_a^{II}(N - n_1 - n_2)]p}{N}$$

ATI (Average Total Inspection) Curve

$ATI = n_1 P_a^I + (n_1 + n_2)P_a^{II} + N(1 - P_a)$

For $P_a = P_a^I + P_a^{II}$

Dodge Romig

I_s = mean number of items inspected per lot of process average quality

$$I_s = N - (N - n) \cdot L(\bar{p}; n, c)$$

Under the condition

$\max_{0<p<1} AOQ\ (p) = p_L$ (AOQL single sampling plans)

Under the condition

$L(p_t; n, c) = 0.10$ (LTPD single sampling plans)

where

$\bar{p}$ = process average fraction defective

N = number of items in the lot

p_t = lot tolerance fraction defective (LTPD)

p_L = average outgoing quality limit (AOQL)

n = number of items in the sample ($n < N$)

c = acceptance number

$L(p)$ = operating characteristic

$AOQ(p)$ = average outgoing quality

Example Dodge Romig plans Single Sampling Table AOQL = 2%

	Process Average, %																	
	0–0.04			0.05–0.40			0.41–0.80			0.81–1.20			1.21–1.60			1.61–2.00		
Lot Size	*n*	*c*	$100p_{0.10}$	*n*	*c*	$100p_{0.10}$	*n*	*c*	$100p_{0.10}$	*n*	*c*	$100p_{0.10}$	*n*	*c*	$100p_{0.10}$	*n*	*c*	$100p_{0.10}$
1–15	All	0	-	All	0	-	All	0	-	All	0	-	All	0	-	All	0	-
16–50	14	0	13.6	14	0	13.6	14	0	13.6	14	0	13.6	14	0	13.6	14	0	13.6
51–100	16	0	12.4	16	0	12.4	16	0	12.4	16	0	12.4	16	0	12.4	16	0	12.4
101–200	17	0	12.2	17	0	12.2	17	0	12.2	17	0	12.2	35	1	10.5	35	1	10.5
201–300	17	0	12.3	17	0	12.3	17	0	12.3	37	1	10.2	37	1	10.2	37	1	10.2
301–400	18	0	11.9	18	0	11.9	38	1	10.0	38	1	10.0	38	1	10.0	60	2	8.5
401–500	18	0	11.9	18	0	11.9	39	1	9.8	39	1	9.8	60	2	8.6	60	2	6.8
501–600	18	0	11.9	18	0	11.9	39	1	9.8	39	1	9.8	60	2	8.6	60	2	8.6
601–800	18	0	11.9	40	1	9.6	40	1	9.6	65	2	8.0	65	2	8.0	85	3	7.5
801–1000	18	0	12.0	40	1	9.6	40	1	9.6	65	2	8.1	65	2	8.1	90	3	7.4
1001–2000	18	0	12.0	41	1	9.4	65	2	8.2	65	2	8.2	95	3	7.0	120	4	6.5
2001–3000	18	0	12.0	41	1	9.4	65	2	8.2	95	3	7.0	120	4	6.5	180	6	5.8
3001–4000	18	0	12.0	42	1	9.3	65	2	8.2	95	3	7.0	155	5	6.0	210	7	5.5
4001–5000	18	0	12.0	42	1	9.3	70	2	7.5	125	4	6.4	155	5	6.0	245	8	5.3
5001–7000	18	0	12.0	42	1	9.3	95	3	7.0	125	4	6.4	185	6	5.6	280	9	5.1
7001–10000	42	1	9.3	70	2	7.5	95	3	7.0	155	5	6.0	220	7	5.4	350	11	4.8
10001–20000	42	1	9.3	70	2	7.6	95	3	7.0	190	6	5.6	290	9	4.9	460	14	4.4
20001–50000	42	1	9.3	70	2	7.6	125	4	6.4	220	7	5.4	395	12	4.5	720	21	3.9
50001–100000	42	1	9.3	95	3	7.0	160	5	5.9	290	9	4.9	505	15	4.2	955	27	3.7

LTPD Single Sampling

Dodge Romig Single Sample Lot Inspection Table

Lot Size	Process Average, %																	
	0–0.02			0.03–0.20			0.21–0.40			0.41–0.60			0.61–0.80			0.81–1.00		
	n	*c*	$100p_{0.10}$	*n*	*c*	$100p_{0.10}$	*n*	*c*	$100p_{0.10}$	*n*	*c*	$100p_{0.10}$	*n*	*c*	$100p_{0.10}$	*n*	*c*	$100p_{0.10}$
1–75	All	0	0.00	All	0	0.00	All	0	0.00	All	0	0.00	All	0	0.00	All	0	0.00
76–100	70	0	0.16	70	0	0.16	70	0	0.16	70	0	0.16	70	0	0.16	70	0	0.16
101–200	85	0	0.25	85	0	0.25	85	0	0.25	85	0	0.25	85	0	0.25	85	0	0.25
201–300	95	0	0.26	95	0	0.26	95	0	0.26	95	0	0.26	95	0	0.26	95	0	0.26
301–400	100	0	0.28	100	0	0.28	100	0	0.28	160	1	0.32	160	1	0.32	160	1	0.32
401–500	105	0	0.28	105	0	0.28	105	0	0.28	165	1	0.34	165	1	0.34	165	1	0.34
501–600	110	0	0.29	110	0	0.29	175	1	0.34	175	1	0.34	175	1	0.34	235	2	0.36
601–800	110	0	0.29	110	0	0.29	180	1	0.36	240	2	0.40	240	2	0.40	300	3	0.41
801–1000	115	0	0.28	115	0	0.28	185	1	0.37	245	2	0.42	305	3	0.44	305	3	0.44
1001–2000	115	0	0.30	190	1	0.40	255	2	0.47	325	3	0.50	380	4	0.54	440	5	0.56
2001–3000	115	0	0.31	190	1	0.41	260	2	0.48	385	4	0.58	450	5	0.60	565	7	0.64
3001–4000	115	0	0.31	195	1	0.41	330	3	0.54	450	5	0.63	510	6	0.65	690	9	0.70
4001–5000	195	1	0.41	260	2	0.50	335	3	0.54	455	5	0.63	575	7	0.69	750	10	0.74
5001–7000	195	1	0.42	265	2	0.50	335	3	0.55	515	6	0.69	640	8	0.73	870	12	0.80
7001–10000	195	1	0.42	265	2	0.50	395	4	0.62	520	6	0.69	760	10	0.79	1050	15	0.86
10001–20000	200	1	0.42	265	2	0.51	460	5	0.67	650	8	0.77	885	12	0.86	1230	18	0.94
20001–50000	200	1	0.42	335	3	0.58	520	6	0.73	710	9	0.81	1060	15	0.93	1520	23	1.00
50001–100000	200	1	0.42	335	3	0.58	585	7	0.76	770	10	0.84	1180	117	0.97	1690	26	1.10

Based on Stated Value of Lot Tolerance per Cent Defective (LTPD)=2.0% and Consumer's Risk = 0.10.

n : Size of Sample; entry of "All" indicates that each piece in lot is to be inspected.
c : Allowable Defect Number for Sample
AOQL : Average Outgoing Quality Limit

Design of Experiments

Experimental Design Equations

Let a "dot" subscript indicate summation over the subscript. Thus:

$$y_{i\bullet} = \sum_{j=1}^{n} y_{ij} \quad \text{and} \quad y_{\bullet\bullet} = \sum_{i=1}^{a} \sum_{j=1}^{n} y_{ij}$$

One-Way Analysis of Variance (ANOVA)

Given independent random samples of size n_i from k populations, then:

$$\sum_{i=1}^{k} \sum_{j=1}^{n_i} \left(y_{ij} - \overline{y}_{\bullet\bullet}\right)^2$$

$$= \sum_{i=1}^{k} n_i\left(\overline{y}_{i\bullet} - \overline{y}_{\bullet\bullet}\right)^2 + \sum_{i=1}^{k} \sum_{i=1}^{n_i} \left(y_{ij} - \overline{y}_{i\bullet}\right)^2$$

$$SS_{\text{total}} = SS_{\text{treatments}} + SS_{\text{error}}$$

If N = total number observations

$$N = \sum_{i=1}^{k} n_i, \ then$$

$$SS_{\text{total}} = \sum_{i=1}^{k} \sum_{j=1}^{n_i} y_{ij}^2 - \frac{y_{\bullet\bullet}^2}{N}$$

$$SS_{\text{treatments}} = \sum_{i=1}^{k} \frac{y_{i\bullet}^2}{n_i} - \frac{y_{\bullet\bullet}^2}{N}$$

$$SS_{\text{error}} = SS_{\text{total}} - SS_{\text{treatments}}$$

From Montgomery, Douglas C., and George C. Runger, *Applied Statistics and Probability for Engineers*, 4th ed., Wiley, 2007.

Randomized Complete Block Design

For *k* treatments and *b* blocks

$$\sum_{i=1}^{k}\sum_{j=1}^{b}\left(y_{ij}-\overline{y}_{\bullet\bullet}\right)^2 = b\sum_{i=1}^{k}\left(\overline{y}_{i\bullet}-\overline{y}_{\bullet\bullet}\right)^2 + \mathrm{k}\sum_{j=1}^{b}\left(\overline{y}_{\bullet j}-\overline{y}_{\bullet\bullet}\right)^2$$
$$+\sum_{i=1}^{k}\sum_{j=1}^{b}\left(\overline{y}_{ij}-\overline{y}_{\bullet j}-\overline{y}_{i\bullet}+\overline{y}_{\bullet\bullet}\right)^2$$
$$SS_{\text{total}} = SS_{\text{treatments}} + SS_{\text{blocks}} + SS_{\text{error}}$$
$$SS_{\text{total}} = \sum_{i=1}^{k}\sum_{j=1}^{b} y_{ij}^2 - \frac{y_{\bullet\bullet}^2}{kb}$$
$$SS_{\text{treatments}} = \frac{1}{b}\sum_{i=1}^{k} y_{i\bullet}^2 - \frac{y_{\bullet\bullet}^2}{bk}$$
$$SS_{\text{blocks}} = \frac{1}{k}\sum_{j=1}^{b} y_{\bullet j}^2 - \frac{y_{\bullet\bullet}^2}{bk}$$
$$SS_{\text{error}} = SS_{\text{total}} - SS_{\text{treatments}} - SS_{\text{blocks}}$$

From Montgomery, Douglas C., and George C. Runger, *Applied Statistics and Probability for Engineers*, 4th ed., Wiley, 2007.

Two-factor Factorial Designs

For *a* levels of Factor A, *b* levels of Factor B, and *n* repetitions per cell:

$$\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}\left(y_{ijk}-\overline{y}_{\bullet\bullet\bullet}\right)^2 = bn\sum_{i=1}^{a}\left(\overline{y}_{i\bullet\bullet}-\overline{y}_{\bullet\bullet\bullet}\right)^2 + an\sum_{j=1}^{b}\left(\overline{y}_{\bullet j\bullet}-\overline{y}_{\bullet\bullet\bullet}\right)^2$$
$$+n\sum_{i=1}^{a}\sum_{j=1}^{b}\left(\overline{y}_{ij\bullet}-\overline{y}_{i\bullet\bullet}-\overline{y}_{\bullet j\bullet}+\overline{y}_{\bullet\bullet\bullet}\right)^2 + \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}\left(y_{ijk}-\overline{y}_{ij\bullet}\right)^2$$
$$SS_{\text{total}} = SS_{\text{A}} + SS_{\text{B}} + SS_{\text{AB}} + SS_{\text{error}}$$
$$SS_{\text{total}} = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n} y_{ijk}^2 - \frac{y_{\bullet\bullet\bullet}^2}{abn}$$
$$SS_{\text{A}} = \sum_{i=1}^{a}\frac{y_{i\bullet\bullet}^2}{bn} - \frac{y_{\bullet\bullet\bullet}^2}{abn}$$
$$SS_{\text{B}} = \sum_{j=1}^{b}\frac{y_{\bullet j\bullet}^2}{an} - \frac{y_{\bullet\bullet\bullet}^2}{abn}$$
$$SS_{\text{AB}} = \sum_{i=1}^{a}\sum_{j=1}^{b}\frac{y_{ij\bullet}^2}{n} - \frac{y_{\bullet\bullet\bullet}^2}{abn} - SS_{\text{A}} - SS_{\text{B}}$$
$$SS_{\text{error}} = SS_{\text{T}} - SS_{\text{A}} - SS_{\text{B}} - SS_{\text{AB}}$$

From Montgomery, Douglas C., and George C. Runger, *Applied Statistics and Probability for Engineers*, 4th ed., Wiley, 2007.

Experimental Design Tables

One-Way ANOVA Table

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	*F*
Between Treatments	$k - 1$	$SS_{\text{treatments}}$	$MST = \frac{SS_{\text{treatments}}}{k-1}$	$\frac{MST}{MSE}$
Error	$N - k$	SS_{error}	$MSE = \frac{SS_{\text{error}}}{N-k}$	
Total	$N - 1$	SS_{total}		

Randomized Complete Block ANOVA Table

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	*F*
Between Treatments	$k - 1$	$SS_{\text{treatments}}$	$MST = \frac{SS_{\text{treatments}}}{k-1}$	$\frac{MST}{MSE}$
Between Blocks	$n - 1$	SS_{blocks}	$MSB = \frac{SS_{\text{blocks}}}{n-1}$	$\frac{MSB}{MSE}$
Error	$(k - 1)(n - 1)$	SS_{error}	$MSE = \frac{SS_{\text{error}}}{(k-1)(n-1)}$	
Total	$N - 1$	SS_{total}		

Two-Way Factorial ANOVA Table

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	*F*
A Treatments	$a - 1$	SS_A	$MSA = \frac{SS_A}{a-1}$	$\frac{MSA}{MSE}$
B Treatments	$b - 1$	SS_B	$MSB = \frac{SS_B}{b-1}$	$\frac{MSB}{MSE}$
AB Interaction	$(a - 1)(b - 1)$	SS_{AB}	$MSAB = \frac{SS_{AB}}{(a-1)(b-1)}$	$\frac{MSAB}{MSE}$
Error	$ab(n - 1)$	SS_{error}	$MSE = \frac{SS_E}{ab(n-1)}$	
Total	$abn - 1$	SS_{total}		

2^n Factorial Experiments

Factors: $X_1, X_2, \ldots, X_n$

Levels of each factor: 1, 2 (sometimes these levels are represented by the symbols – and +, respectively)

r = number of observations for each experimental condition (treatment)

E_i = estimate of the effect of factor X_i, $i = 1, 2, \ldots, n$

E_{ij} = estimate of the effect of the interaction between factors X_i and X_j

$\bar{Y}_{ik}$ = average response value for all $r2^{n-1}$ observations having X_i set at level k, $k = 1, 2$

$\bar{Y}_{ij}^{km}$ = average response value for all $r2^{n-2}$ observations having X_i set at level k, $k = 1, 2$, and X_j set at level m, $m = 1, 2$.

$$E_i = \overline{Y}_{i2} - \overline{Y}_{i1}$$

$$E_{ij} = \frac{\left(\overline{Y}_{ij}^{22} - \overline{Y}_{ij}^{21}\right) - \left(\overline{Y}_{ij}^{12} - \overline{Y}_{ij}^{11}\right)}{2}$$

Analysis of Variance for 2^n Factorial Designs

Main Effects

Let E be the estimate of the effect of a given factor, let L be the orthogonal contrast belonging to this effect. It can be proved that

$$E = \frac{L}{2^{n-1}}$$

$$L = \sum_{c=1}^{m} a_{(c)} \overline{Y}_{(c)}$$

$$SS_L = \frac{rL^2}{2^n}, \text{ where}$$

m = number of experimental conditions ($m = 2^n$ for n factors)

$a_{(c)} = -1$ if the factor is set at its low level (level 1) in experimental condition c

$a_{(c)} = +1$ if the factor is set at its high level (level 2) in experimental condition c

r = number of replications for each experimental condition

$\bar{Y}_{(c)}$ = average response value for experimental condition c

SS_L = sum of squares associated with the factor

Interaction Effects

Consider any group of two or more factors.

$a_{(c)} = +1$ if there is an even number (or zero) of factors in the group set at the low level (level 1) in experimental condition $c = 1, 2, \ldots, m$

$a_{(c)} = -1$ if there is an odd number of factors in the group set at the low level (level 1) in experimental condition $c = 1, 2, \ldots, m$

It can be proved that the interaction effect E for the factors in the group and the corresponding sum of squares SS_L can be determined as follows:

$$E = \frac{L}{2^{n-1}}$$

$$L = \sum_{c=1}^{m} a_{(c)} \overline{Y}(c)$$

$$SS_L = \frac{rL^2}{2^n}$$

Sum of Squares of Random Error

The sum of the squares due to the random error can be computed as

$$SS_{\text{error}} = SS_{\text{total}} - \Sigma_i SS_i - \Sigma_i \Sigma_j SS_{ij} - \ldots - SS_{12\ldots n}$$

where SS_i is the sum of squares due to factor X_i, SS_{ij} is the sum of squares due to the interaction of factors X_i and X_j, and so on. The total sum of squares is equal to

$$SS_{\text{total}} = \sum_{c=1}^{m} \sum_{k=1}^{r} Y_{ck}^2 - \frac{T^2}{N}$$

where Y_{ck} is the kth observation taken for the cth experimental condition, $m = 2^n$, T is the grand total of all observations, and $N = r2^n$.

Taguchi Loss Function

$$L = k(y - m)^2$$

where

L = monetary loss ($)

k = loss coefficient

y = quality characteristic (metric/kpi)

m = target value for y

Failure Modes and Effects Analysis Template

FMEA Format

Function	Potential Failure Mode	Potential Effect(s) of Failure	S	Class	Potential Cause(s)/ Mechanism(s) of Failure	O	Current Controls	D	RPN	Recommended Action(s)	Responsibility and Target Completion Date	Action Results				
												Actions Taken	S	O	D	RPN

where

RPN = Risk Priority Number

S = Severity, usually rated on a scale from 1 to 10, where 1 is insignificant and 10 is catastrophic.

O = Occurrence rating, usually rated on a scale from 1 to 10, where 1 is extremely unlikely and 10 is inevitable.

D = Detection rating, usually rated on a scale from 1 to 10, where 1 means the control is absolutely certain to detect the problem and 10 means
the control is certain not to detect the problem (or no control exists).

RPN = S x O x D

Criticality = S x O

Reliability Analysis

System Reliability Models

Reliability is the cumulative probability function of success. Reliability at time, t, is defined as:

$$R(t) = \frac{n_s(t)}{n_s(t) + n_f(t)}$$

where

t = time

$n_s(t)$ = number of surviving components during time interval

$n_f(t)$ = number of failed components during time interval

For an exponential distribution, reliability can also be defined as:

$$R(t) = e^{-\lambda t}$$

where

λ = failure rate

t = time

If P_i is the probability that a component i is functioning, a reliability function $R(P_1, P_2, \dots, P_n)$ represents the probability that a system will work.

For n independent components connected in series,

$$R(P_1, P_2, \dots, P_n) = \prod_{i=1}^{n} P_i$$

For n independent components connected in parallel,

$$R(P_1, P_2, \dots, P_n) = 1 - \prod_{i=1}^{n} (1 - P_i)$$

Maintainability

T = time to restore for t

t = time to failure

μ = repair rate for λ

λ = failure rate

$P(T \le t)$ for $F(t)$ probability of failing by age t

$f(t)$ = time to failure, probability density function

$g(t)$ = time to repair, probability density function

$R(t)$ = reliability function

$M(t)$ = maintainability function

$\lambda(t)$ = failure rate function

$\mu(t)$ = repair rate function

MTTF = mean time to failure

MTTR = mean time to repair

Lognormal Distribution

$$g(t = M_{ct_i}) = \frac{1}{M_{ct_i} S_{lnM_{ct}} \sqrt{2\pi}} e^{\left[-\frac{\left(ln\left(M_{ct_i}\right) - \overline{ln(M_{ct})}\right)^2}{2(S_{lnM_{ct}})^2}\right]}$$

$$g(t = M_{ct_i}) = \frac{1}{t\sigma_{t'}\sqrt{2\pi}} e^{\left[-\frac{1}{2}\left(\frac{t' - \bar{t'}}{\sigma_{t'}}\right)^2\right]}$$

where

$t = M_{ct_i}$ = repair time from each failure

$S_{lnM_{ct}}$ = standard deviation of the natural logarithm of the repair times

$t' = \ln(M_{ct_i}) = \ln(t)$

N = number of repair actions

$$\overline{ln(M_{ct})} = \frac{\sum ln\left(M_{ct_i}\right)}{N}$$

$$S_{lnM_{ct}} = \sigma_{t'} = \sqrt{\frac{\sum ln\left(M_{ct_i}\right)^2 - \frac{ln\left(M_{ct_i}\right)^2}{N}}{N-1}}$$

$$S_{lnM_{ct}} = \sqrt{\frac{\sum {t'_i}^2 - \frac{\left(\sum t'_i\right)^2}{N}}{N-1}}$$

$$\bar{t'} = \overline{ln(M_{ct})} = \frac{\sum t'_i}{N}$$

$S_{lnM_{ct}}$ = standard deviation of the natural logarithm of the repair time

$t' = \ln(M_{ct_i}) = \ln(t)$

Mean Time to Repair

$$MTTR = \overline{M_{ct}} = \bar{t} = \int_0^\infty t\ g(t = M_{ct_i})dt$$

$$MTTR = \overline{M_{ct}} = \bar{t} = e^{\left[\overline{ln(M_{ct})} + \frac{1}{2}\left(S_{ln(M_{ct})}\right)^2\right]}$$

$$MTTR = \overline{M_{ct}} = \bar{t} = e^{\left[\overline{t'} + \frac{1}{2}(\sigma_{t'})^2\right]}$$

Median Time to Repair

$$\widetilde{M_{ct}} = \frac{\sum \lambda_i\ \overline{ln(M_{ct})}}{\sum \lambda_i}$$

$$\widetilde{M_{ct}} = \overline{ln(M_{ct})}$$

$$\widetilde{M_{ct}} = e^{\overline{t'}}$$

Maximum Time to Repair

The maximum time to repair is given by

$$M_{\max ct} = t_{max} = e^{\left(\overline{ln(M_{ct})} + \phi\, S_{ln(M_{ct})}\right)}$$

$$M_{\max ct} = t_{max} = e^{\left(\overline{t'} + Z(t'_{1-\alpha})\,\sigma\, t'\right)}$$

Where $\phi = Z(t'_{1-\alpha})$ is the value from the normal distribution function corresponding to the percentage point (1 – α) on the maintainability function for which $M_{\max ct}$ is defined. The most commonly used values of ϕ or $Z(t_{1-\alpha})$ are:

1-α	Φ or Z ($t_{1-\alpha}$)
0.80	0.8416
0.85	1.036
0.90	1.282
0.95	1.645
0.99	2.326

Normal Distribution

$$g(t = M_{ct}) = \frac{1}{S_{M_{ct}}\sqrt{2\pi}} e^{\left[\frac{-(M_{ct_i} - \bar{M}_{ct})^2}{2(SM_{ct})^2}\right]}$$

where
M_{ct_i} = repair time for an individual maintenance action

and the average repair time for *N* observations is:

$$\bar{M}_{ct} = \frac{\sum M_{ct_i}}{N}$$

The standard deviation of the distribution of repair times, based on *N* observations is

$$S_{M_{ct}} = \sqrt{\frac{\sum(M_{ct_i} - \bar{M}_{ct})^2}{N-1}}$$

The mean time to repair is given by

$$\bar{M}_{ct} = \frac{\sum M_{ct_i}}{N}$$

The median time to repair is given by

$$\tilde{M}_{ct} = \frac{\sum M_{ct_i}}{N}$$

The maximum time to repair is given by

$$\tilde{M}_{MAXct} = \bar{M}_{ct} + \phi S_{\bar{M}_{ct}}$$

where $\phi = Z(t'_{1-\alpha})$ is the value from the normal distribution function corresponding to the percentage point (1–α) on the maintainability function for which $M_{max\ ct}$ is defined. The most commonly used values of ϕ or $Z(t_{t-\alpha})$ are:

1-α	Φ or Z ($t_{1-\alpha}$)
0.80	0.8416
0.85	1.036
0.90	1.282
0.95	1.645
0.99	2.326

Exponential Distribution

$$g(t = M_{ct}) = \frac{1}{M_{ct}} e^{-\frac{M_{ct_i}}{\bar{M}_{ct}}}$$

$$g(t) = \mu e^{-\mu t}$$

$$M(t) = \int_0^t g(t)dt = \int_0^t \mu e^{-\mu t} dt = 1 - e^{-\mu t}$$

$$\bar{M}_{ct} = \frac{1}{\mu} = \frac{\sum M_{ct_i}}{N}$$

$$MTTR = \bar{M}_{ct} = \frac{-t}{\ln(1 - M(t))}$$

$$\tilde{M}_{ct} = 0.69\,\bar{M}_{ct}$$

The standard deviation of the distribution of repair times, based on *N* observations is:

$$S_{M_{ct}} = \sqrt{\frac{\sum(M_{ct_i} - \bar{M}_{ct})^2}{N-1}}$$

The mean time to repair is given by

$$\bar{M}_{ct} = \frac{\sum M_{ct_i}}{N}$$

The median time to repair is given by

$$\tilde{M}_{ct} = \frac{\sum M_{ct_i}}{N}$$

which is equal to the mean time to repair because of the symmetry of the normal distribution.

The maximum time to repair is given by

$$\tilde{M}_{MAXct} = \bar{M}_{ct} + \phi S_{\bar{M}_{ct}}$$

where $\phi = Z(t'_{1-\alpha})$ is the value from the normal distribution function corresponding to the percentage point (1–α) on the maintainability function for which $M_{max\,ct}$ is defined. The most commonly used values of ϕ or $Z(t_{1-\alpha})$ are:

1-α	Φ or Z ($t_{1-\alpha}$)
0.80	0.8416
0.85	1.036
0.90	1.282
0.95	1.645
0.99	2.326

Cumulative Failure Rate

$\Lambda(t) = kt^{-\beta}$

where

$\Lambda(t)$ = cumulative failure rate at time t

t = cumulative test time

k = a constant ($k > 0$)

β = growth factor ($\beta > 0$)

Actual Failure Rate

Actual failure rate of the system, if design is released after test time t is given by:

$\lambda(t) = k(1 - \beta)t^{-\beta}$

where

k = a constant ($k > 0$)

β = growth factor ($\beta > 0$)

Time to Failure Distribution

$f(t) = \lambda e^{-\lambda t}$

where

λ = failure rate

t = time

Mean Time to Failure (MTTF)

For a constant hazard rate model, the mean time to failure is:

$$MTTF = \frac{1}{\lambda}$$

where

λ = failure rate

The Hazard Function

Conditional probability of failing in the next small interval given survival up to time t.

$$h(t) = \frac{f(t)}{1 - F(t)} = \frac{f(t)}{R(t)}$$

where

$f(t)$ = probability density function (PDF)

$F(t)$ = cumulative distribution function (CDF)

$R(t)$ = reliability function, also referred to as the survival function

Availability Measures

Achieved Availability (A_a)

$$\mathrm{A_a} = \frac{\mathrm{OT}}{\mathrm{OT} + \mathrm{TPM} + \mathrm{TCM}}$$

where

OT = operating time

TPM = total preventative maintenance time

TCM = total corrective maintenance time

Operation Availability (A_o)

$$\mathrm{A_o} = \frac{\mathrm{OT} + \mathrm{ST}}{\mathrm{OT} + \mathrm{ST} + \mathrm{TPM} + \mathrm{TCM} + \mathrm{ALDT}}$$

where

OT = operating time
TPM = total preventative maintenance time
TCM = total corrective maintenance time
ST = standby time
ALDT = administrative and logistic delay time

Inherent Availability (A_I)

$$\mathrm{A_I} = \frac{\mathrm{MTBF}}{\mathrm{MTBF} + \mathrm{MTTR}}$$

where

MTTR = mean time to repair

MTBF = mean time between failures

6 General

Units

Currently some of the exam questions use both the metric system of units and the U.S. Customary System (USCS). In the USCS system of units, both force and mass are called pounds. Therefore, one must distinguish the pound-force (lbf) from the pound-mass (lbm).

The pound-force is that force which accelerates one pound-mass at 32.174 ft/sec^2. Thus, 1 lbf = 32.174 lbm-ft/sec^2. The expression 32.174 lbm-ft/(lbf-sec^2) is designated as g_c and is used to resolve expressions involving both mass and force expressed as pounds. For instance, in writing Newton's second law, the equation would be written as $F = ma/g_c$, where F is in lbf, m in lbm, and a is in ft/sec^2.

Similar expressions exist for other quantities: Kinetic Energy: $KE = mv^2/2g_c$, with KE in (ft-lbf); Potential Energy: $PE = mgh/g_c$, with PE in (ft-lbf); Fluid Pressure: $p = \rho gh/g_c$, with p in (lbf/ft^2); Specific Weight: $SW = \rho g/g_c$, in (lbf/ft^3); Shear Stress: $\tau = (\mu/g_c)(dv/dy)$, with shear stress in (lbf/ft^2). In all these examples, g_c should be regarded as a force unit conversion factor. It is frequently not written explicitly in engineering equations. However, its use is required to produce a consistent set of units.

Note that the conversion factor g_c [lbm-ft/(lbf-sec^2)] should not be confused with the local acceleration of gravity g, which has different units (m/s^2 or ft/sec^2) and may be either its standard value (9.807 m/s^2 or 32.174 ft/sec^2) or some other local value.

If the problem is presented in USCS units, it may be necessary to use the constant g_c in the equation to have a consistent set of units.

Metric Prefixes, Commonly Used Equivalents, and Temperature Conversions

METRIC PREFIXES		
Multiple	**Prefix**	**Symbol**
10^{-18}	atto	a
10^{-15}	femto	f
10^{-12}	pico	p
10^{-9}	nano	n
10^{-6}	micro	μ
10^{-3}	milli	m
10^{-2}	centi	c
10^{-1}	deci	d
10^{1}	deka	da
10^{2}	hecto	h
10^{3}	kilo	k
10^{6}	mega	M
10^{9}	giga	G
10^{12}	tera	T
10^{15}	peta	P
10^{18}	exa	E

COMMONLY USED EQUIVALENTS	
1 gallon of water weighs	8.34 lbf
1 cubic foot of water weighs	62.4 lbf
1 cubic inch of mercury weighs	0.491 lbf
The mass of 1 cubic meter of water is	1,000 kg
1 mg/L is	8.34×10^{-6} lbf/gal

TEMPERATURE CONVERSIONS

$$°F = 1.8\,(°C) + 32$$

$$°C = \frac{°F - 32}{1.8}$$

$$°R = °F + 459.69$$

$$K = °C + 273.15$$

Fundamental Constants

Quantity		Symbol	Value	Units
electron charge		e	1.6022×10^{-19}	C (coulombs)
Faraday constant		F	96,485	coulombs/(mol)
gas constant	metric	$\bar{R}$	8,314	J/(kmol·K)
gas constant	metric	$\bar{R}$	8.314	kPa·m^3/(kmol·K)
gas constant	USCS	$\bar{R}$	1,545	ft-lbf/(lb mole-°R)
		$\bar{R}$	0.08206	L-atm/mole-K
gravitation - newtonian constant		G	6.673×10^{-11}	$m^3/(kg \cdot s^2)$
gravitation - newtonian constant		G	6.673×10^{-11}	$N \cdot m^2/kg^2$
gravity acceleration (standard)	metric	g	9.807	m/s^2
gravity acceleration (standard)	USCS	g	32.174	ft/sec^2
molar volume (ideal gas), T = 273.15K, p = 101.3 kPa		V_m	22,414	L/kmol
speed of light in vacuum		c	299,792,000	m/s

Conversion Factors

Multiply	By	To Obtain	Multiply	By	To Obtain
acre	43,560	square feet (ft^2)	joule (J)	9.478×10^{-4}	Btu
ampere-hr (A-hr)	3,600	coulomb (C)	J	0.7376	ft-lbf
ångström (Å)	1×10^{-10}	meter (m)	J	1	newton·m (N·m)
atmosphere (atm)	76.0	cm, mercury (Hg)	J/s	1	watt (W)
atm, std	29.92	in, mercury (Hg)			
atm, std	14.70	lbf/in^2 abs (psia)	kilogram (kg)	2.205	pound (lbm)
atm, std	33.90	ft, water	kgf	9.8066	newton (N)
atm, std	1.013×10^{5}	pascal (Pa)	kilometer (km)	3,281	feet (ft)
			km/hr	0.621	mph
bar	1×10^{5}	Pa	kilopascal (kPa)	0.145	lbf/in^2 (psi)
barrels–oil	42	gallons–oil	kilowatt (kW)	1.341	horsepower (hp)
Btu	1,055	joule (J)	kW	3,413	Btu/hr
Btu	2.928×10^{-4}	kilowatt-hr (kWh)	kW	737.6	(ft-lbf)/sec
Btu	778	ft-lbf	kW-hour (kWh)	3,413	Btu
Btu/hr	3.930×10^{-4}	horsepower (hp)	kWh	1.341	hp-hr
Btu/hr	0.293	watt (W)	kWh	3.6×10^{6}	joule (J)
Btu/hr	0.216	ft-lbf/sec	kip (K)	1,000	lbf
			K	4,448	newton (N)
calorie (g-cal)	3.968×10^{-3}	Btu			
cal	1.560×10^{-6}	hp-hr	liter (L)	61.02	in^3
cal	4.186	joule (J)	L	0.264	gal (US Liq)
cal/sec	4.186	watt (W)	L	10^{-3}	m^3
centimeter (cm)	3.281×10^{-2}	foot (ft)	L/second (L/s)	2.119	ft^3/min (cfm)
cm	0.394	inch (in)	L/s	15.85	gal (US)/min (gpm)
centipoise (cP)	0.001	pascal·sec (Pa·s)			
centistokes (cSt)	1×10^{-6}	m^2/sec (m^2/s)	meter (m)	3.281	feet (ft)
cubic feet/second (cfs)	0.646317	million gal/day (MGD)	m	1.094	yard
cubic foot (ft^3)	7.481	gallon	m/second (m/s)	196.8	feet/min (ft/min)
cubic meters (m^3)	1,000	liters	mile (statute)	5,280	feet (ft)
electronvolt (eV)	1.602×10^{-19}	joule (J)	mile (statute)	1.609	kilometer (km)
			mile/hour (mph)	88.0	ft/min (fpm)
foot (ft)	30.48	cm	mph	1.609	km/h
ft	0.3048	meter (m)	mm of Hg	1.316×10^{-3}	atm
ft-pound (ft-lbf)	1.285×10^{-3}	Btu	mm of H_2O	9.678×10^{-5}	atm
ft-lbf	3.766×10^{-7}	kilowatt-hr (kWh)			
ft-lbf	0.324	calorie (g-cal)	newton (N)	0.225	lbf
ft-lbf	1.356	joule (J)	N·m	0.7376	ft-lbf
ft-lbf/sec	1.818×10^{-3}	horsepower (hp)	N·m	1	joule (J)
gallon (US Liq)	128	fluid ounces			
gallon (US Liq)	3.785	liter (L)	pascal (Pa)	9.869×10^{-6}	atmosphere (atm)
gallon (US Liq)	0.134	ft^3	Pa	1	newton/m^2 (N/m^2)
gallons of water	8.3453	pounds of water	Pa·sec (Pa·s)	10	poise (P)
gamma (γ, Γ)	1×10^{-9}	tesla (T)	pound (lbm,avdp)	0.454	kilogram (kg)
gauss	1×10^{-4}	T	lbf	4.448	N
gram (g)	2.205×10^{-3}	pound (lbm)	lbf-ft	1.356	N·m
			lbf/in^2 (psi)	0.068	atm
hectare	1×10^{4}	square meters (m^2)	psi	2.307	ft of H_2O
hectare	2.47104	acres	psi	2.036	in of Hg
horsepower (hp)	42.4	Btu/min	psi	6,895	Pa
hp	745.7	watt (W)			
hp	33,000	(ft-lbf)/min	radian	$180/\pi$	degree
hp	550	(ft-lbf)/sec			
hp-hr	2,544	Btu	stokes	1×10^{-4}	m^2/s
hp-hr	1.98×10^{6}	ft-lbf			
hp-hr	2.68×10^{6}	joule (J)	therm	1×10^{5}	Btu
hp-hr	0.746	kWh			
			watt (W)	3.413	Btu/hr
inch (in)	2.540	centimeter (cm)	W	1.341×10^{-3}	horsepower (hp)
in of Hg	0.0334	atm	W	1	joule/sec (J/s)
in of Hg	13.60	in of H_2O	weber/m^2 (Wb/m^2)	10,000	gauss
in of H_2O	0.0361	lbf/in^2 (psi)			
in of H_2O	0.002458	atm			

Probability and Statistics

Permutations and Combinations

A *permutation* is a particular sequence of a given set of objects. A *combination* is the set itself without reference to order.

1. The number of different *permutations* of n distinct objects *taken* r *at a time* is

$$P(n,r) = \frac{n!}{(n-r)!}$$

   nPr is an alternative notation for $P(n,r)$

2. The number of different *combinations* of n distinct objects *taken* r *at a time* is

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{[r!\,(n-r)!]}$$

   nCr and $\binom{n}{r}$ are alternative notations for $C(n,r)$

3. The number of different *permutations* of n objects *taken* n *at a time*, given that n_i are of type i, where $i = 1, 2, \ldots, k$ and $\Sigma n_i = n$, is

$$P(n; n_i, n_2, \ldots, n_k) = \frac{n!}{n_1!\, n_2! \ldots n_k!}$$

Laws of Probability

Property 1. General Character of Probability

The probability $P(E)$ of an event E is a real number in the range of 0 to 1. The probability of an impossible event is 0 and that of an event certain to occur is 1.

Property 2. Law of Total Probability

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

where

$P(A \cup B)$ = the probability that either A or B occur alone or that both occur together
$P(A)$ = the probability that A occurs
$P(B)$ = the probability that B occurs
$P(A \cap B)$ = the probability that both A and B occur simultaneously

Property 3. Law of Compound or Joint Probability

If neither $P(A)$ nor $P(B)$ is zero,

$$P(A \cap B) = P(A)P(B|A) = P(B)P(A|B)$$

where

$P(B|A)$ = the probability that B occurs given the fact that A has occurred
$P(A|B)$ = the probability that A occurs given the fact that B has occurred
$P(A \cap B)$ = the probability that both A and B occur simultaneously

If either $P(A)$ or $P(B)$ is zero, then $P(A \cap B) = 0$

Bayes' Theorem

$$P(Bj|A) = \frac{P(B_j)P(A|B_j)}{\sum_{i=1}^{n} P(A|B_i)P(B_i)}$$

where

$P(A_j)$ = the probability of event A_j within the population of A
$P(B_j)$ = the probability of event B_j within the population of B

Probability Functions

A random variable X has a probability associated with each of its values. The probability is termed a discrete probability if X can assume only the discrete values, or

$$X = x_1, x_2, x_3, \ldots, x_n$$

The *discrete probability* of the event $X = x_i$ occurring is defined as $P(x_i)$ while the *probability mass function* of the random variable X is defined by

$$f(x_k) = P(X = x_k), k = 1, 2, \ldots, n$$

Probability Density Functions

If X is continuous, then the *probability density function* $f(x)$ is defined so that

$$P(a \leq X \leq b) = \int_a^b f(x)dx$$

Cumulative Distribution Functions

The *cumulative distribution function*, *F,* of the discrete probability function $P(x_i)$ is defined as

$$F(x_m) = \sum_{k=1}^{m} P(x_k) = P(X \leq x_m), m = 1, 2, \ldots, n$$

If X is continuous, the *cumulative distribution function*, F, is defined by

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

which implies that *F(a)* is the probability that $X \le a$.

Expected Values

Let X be a discrete random variable having a probability mass function

$f(x_k), k = 1, 2, \dots, n$

The expected value of X is defined as

$$\mu = E[X] = \sum_{k=1}^{n} x_k f(x_k)$$

The variance of X is defined as

$$\sigma^2 = V[X] = \sum_{k=1}^{n} (x_k - \mu)^2 f(x_k)$$

Let X be a continuous random variable having a density function $f(X)$ and let $Y = g(X)$ be some general function. The expected value of Y is

$$E[Y] = E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx$$

The mean or expected value of the random variable X is now defined as

$$\mu = E[X] = \int_{-\infty}^{\infty} xf(x)dx$$

While the variance is given by

$$\sigma^2 = V[X] = E[(X - \mu)^2] = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)dx$$

The standard deviation is given by

$$\sigma = \sqrt{V[X]}$$

The coefficient of variation is defined as $\frac{\sigma}{\mu}$.

Sums of Random Variables

$Y = a_1 X_1 + a_2 X_2 + \ldots + a_n X_n$

The expected value of *Y* is:

$$\mu_y = E[Y] = a_1 E[X_1] + a_2 E[X_2] + \cdots + a_n E[X_n]$$

If the random variables are statistically *independent*, then the variance of *Y* is:

$$\sigma_y^2 = V[Y] = a_1^2 V[X_1] + a_2^2 V[X_2] + \cdots + a_n^2 V[X_n]$$

$$= a_1^2 \sigma_1^2 + a_2^2 \sigma_2^2 + \cdots + a_n^2 \sigma_n^2$$

Also, the standard deviation of *Y* is:

$$\sigma_y = \sqrt{\sigma_y^2}$$

Normal Distribution (Gaussian Distribution)

This is a unimodal distribution, the mode being $x = \mu$, with two points of inflection (each located at a distance σ to either side of the mode). The averages of *n* observations tend to become normally distributed as *n* increases. The variate *x* is said to be normally distributed if its density function *f* (*x*) is given by an expression of the form:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

where

μ = population mean

σ = standard deviation of the population

$-\infty \le x \le \infty$

When $\mu = 0$ and $\sigma^2 = \sigma = 1$, the distribution is called a *standardized* or *unit normal* distribution. Then

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad \text{where } -\infty \le x \le \infty.$$

The *Z*-transformation formula is:

$$Z = \frac{x - \mu}{\sigma}$$

A unit normal distribution table is included at the end of this section. In the table, the following notations are utilized:

F(*x*) = the area under the curve from $-\infty$ to *x*,

R(*x*) = the area under the curve from *x* to ∞, and

W(*x*) = the area under the curve between $-x$ and *x*.

The Central Limit Theorem

Let $X_1, X_2, \ldots, X_n$ be a sequence of independent and identically distributed random variables each having mean μ and variance σ^2. Then for large n, the Central Limit Theorem asserts that the sum

$Y = X_1 + X_2 + \ldots X_n$ is approximately normal.

$$\mu_{\bar{y}} = \mu$$

and the standard deviation

$$\sigma_{\bar{y}} = \frac{\sigma}{\sqrt{n}}$$

t-Distribution

The variate t is defined as the quotient of two independent variates x and r where *x is unit normal* and *r is the root mean square* of n other independent *unit normal variates*; that is, $t = x/r$. The following is the t-distribution with n degrees of freedom:

$$f(t) = \frac{\Gamma\,[(n+1)]/2}{\Gamma(n/2)\sqrt{n\pi}} \frac{1}{(1 + t^2/n)^{(n+1)/2}}$$

where $-\infty \le t \le \infty$.

A table at the end of this section gives the values of $t_{\alpha,\,n}$ for values of α and n. Note that in view of the symmetry of the t-distribution,

$t_{1-\alpha,n} = -t_{\alpha,n}$.

The function for α follows:

$$\alpha = \int_{t_{\alpha,n}}^{\infty} f(t)dt$$

A table showing probability and density functions is included at the end of this section.

χ^2 - Distribution

If $Z_1, Z_2, \ldots, Z_n$ are independent unit normal random variables, then

$$\chi^2 = Z_1^2 + Z_2^2 + \ldots Z_n^2$$

is said to have a chi-square distribution with n degrees of freedom.
A table at the end of this section gives values of $\chi^2_{\alpha,n}$ for selected values of α and n.

Standard Deviation and Variance

The *standard deviation* of a population is

$$\sigma = \sqrt{(1/N)\sum(X_i - \mu)^2}$$

The *sample variance* is

$$s^2 = [1/(n-1)]\sum_{i=1}^{n}(X_i - \bar{X})^2$$

The *sample standard deviation* is

$$s = \sqrt{\left[\frac{1}{n-1}\right]\sum_{i=1}^{n}(X_i - \bar{X})^2}$$

The *coefficient of variation = CV = s/* $\overline{X}$

Standard deviation of a proportion is

$$\sqrt{\frac{pq}{n}}$$

where

p = probability of occurrence

q = probability of non-occurrence

n = number of observations

Confidence Intervals

Confidence Interval for the Mean μ of a Normal Distribution

(a) Standard deviation σ is known

$$\bar{X} - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{X} + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

(b) Standard deviation σ is not known

$$\bar{X} - t_{\alpha/2}\frac{s}{\sqrt{n}} \le \mu \le \bar{X} + t_{\alpha/2}\frac{s}{\sqrt{n}}$$

where $t_{\alpha/2}$ corresponds to *n* – 1 degrees of freedom.

Confidence Interval for the Difference Between Two Means μ_1 and μ_2

(c) Standard deviations σ_1 and σ_2 known

$$\bar{X}_1 - \bar{X}_2 - Z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \le \mu_1 - \mu_2 \le \bar{X}_1 - \bar{X}_2 + Z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

(d) Standard deviations σ_1 and σ_2 are not known

$$\bar{X}_1 - \bar{X}_2 - t_{\alpha/2}\sqrt{\frac{\left(\frac{1}{n_1}+\frac{1}{n_2}\right)[(n_1-1)s_1^2+(n_2-1)s_2^2]}{n_1+n_2-2}} \le \mu_1 - \mu_2$$

$$\le \bar{X}_1 - \bar{X}_2 + t_{\alpha/2}\sqrt{\frac{\left(\frac{1}{n_1}+\frac{1}{n_2}\right)[(n_1-1)s_1^2+(n_2-1)s_2^2]}{n_1+n_2-2}}$$

where $t_{\alpha/2}$ corresponds to $n_1 + n_2 - 2$ degrees of freedom.

Confidence Intervals for the Variance σ^2 of a Normal Distribution

$$\frac{(n-1)s^2}{x^2_{\alpha/2,\,n-1}} \le \sigma^2 \le \frac{(n-1)s^2}{x^2_{1-\alpha/2,\,n-1}}$$

Sample Size

$$z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \qquad n = \left[\frac{z_{\alpha/2}\sigma}{\overline{x} - \mu}\right]^2$$

Test Statistic

The following definitions apply.

$$Z_{\text{var}} = \frac{\overline{X} - \mu_o}{\frac{\sigma}{\sqrt{n}}}$$

$$t_{\text{var}} = \frac{\overline{X} - \mu_o}{\frac{s}{\sqrt{n}}}$$

where

Z_{var} is the standard normal Z score

t_{var} is the sample distribution test statistic

σ is known standard deviation

μ_o is population mean

$\overline{X}$ is hypothesized mean or sample mean

n is sample size

s is computed sample standard deviation

Values of $Z_{\alpha/2}$

Confidence Interval	$Z_{\alpha/2}$
80%	1.2816
90%	1.6449
95%	1.9600
96%	2.0537
98%	2.3263
99%	2.5758

The Z score is applicable when the standard deviation(s) is known. The test statistic is applicable when the standard deviation(s) is computed at time of sampling.

Z_α corresponds to the appropriate probability under the normal probability curve for a given Z_{var}.

$t_{\alpha,\, n-1}$ corresponds to the appropriate probability under the t distribution with $n - 1$ degrees of freedom for a given t_{var}.

Probability and Density Functions: Means and Variances

Variable	Equation	Mean	Variance
Binomial Coefficient	$\binom{n}{x} = \frac{n!}{x!(n-x)!}$		
Binomial	$b(x;n,p) = \binom{n}{x} p^x (1-p)^{n-x}$	np	$np(1-p)$
Hyper Geometric	$h(x;n,r,N) = \binom{r}{x} \frac{\binom{N-r}{n-x}}{\binom{N}{n}}$	$\frac{nr}{N}$	$\frac{r(N-r)n(N-n)}{N^2(N-1)}$
Poisson	$f(x;\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$	λ	λ
Geometric	$g(x; p) = p\ (1-p)^{x-1}$	$1/p$	$(1-p)/p^2$
Negative Binomial	$f(y;r,p) = \binom{y+r-1}{r-1} p^r (1-p)^y$	r/p	$r\ (1-p)/p^2$
Multinomial	$f(x_1,\ldots x_k) = \frac{n!}{x_1!,\ldots,x_k!} p_1^{x_1} \ldots p_k^{x_k}$	np_i	$np_i\ (1-p_i)$
Uniform	$f(x) = 1/(b-a)$	$(a+b)/2$	$(b-a)^2/12$
Gamma	$f(x) = \frac{x^{\alpha-1} e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)};\quad \alpha > 0, \beta > 0$	$\alpha\beta$	$\alpha\beta^2$
Exponential	$f(x) = \frac{1}{\beta} e^{-x/\beta}$	β	β^2
Weibull	$f(x) = \frac{\alpha}{\beta} x^{\alpha-1} e^{-x^\alpha/\beta}$	$\beta^{1/\alpha}\Gamma[(\alpha+1)/\alpha]$	$\beta^{2/\alpha}\left[\Gamma\left(\frac{\alpha+1}{\alpha}\right) - \Gamma^2\left(\frac{\alpha+1}{\alpha}\right)\right]$
Normal	$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$	μ	σ^2
Standardized or Unit Normal	$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$	0	1
Triangular	$f(x) = \begin{cases} \frac{2(x-a)}{(b-a)(m-a)} & \text{if } a \le x \le m \\ \frac{2(b-x)}{(b-a)(b-m)} & \text{if } m < x \le b \end{cases}$	$\frac{a+b+m}{3}$	$\frac{a^2+b^2+m^2-ab-am-bm}{18}$

Hypothesis Testing

Tests on Means of Normal Distribution - Variance Known

Hypothesis	*Test Statistic*	*Criteria for Rejection*
$H_0: \mu = \mu_0$ $H_1: \mu \neq \mu_0$		$\lvert Z_0 \rvert > Z_{\alpha/2}$
$H_0: \mu = \mu_0$ $H_0: \mu < \mu_0$	$Z_0 \equiv \dfrac{\overline{X} - \mu_0}{\sigma / \sqrt{n}}$	$Z_0 < -Z_\alpha$
$H_0: \mu = \mu_0$ $H_1: \mu > \mu_0$		$Z_0 > Z_\alpha$
$H_0: \mu_1 - \mu_2 = \gamma$ $H_1: \mu_1 - \mu_2 \neq \gamma$		$\lvert Z_0 \rvert > Z_{\alpha/2}$
$H_0: \mu_1 - \mu_2 = \gamma$ $H_1: \mu_1 - \mu_2 < \gamma$	$Z_0 \equiv \dfrac{\overline{X_1} - \overline{X_2} - \gamma}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}}$	$Z_0 <- Z_\alpha$
$H_0: \mu_1 - \mu_2 = \gamma$ $H_1: \mu_1 - \mu_2 > \gamma$		$Z_0 > Z_\alpha$

Tests on Means of Normal Distribution - Variance Unknown

Hypothesis	*Test Statistic*	*Criteria for Rejection*
H_0: $\mu = \mu_0$ H_1: $\mu \neq \mu_0$		$\lvert t_0 \rvert > t_{\alpha/2,\, n-1}$
H_0: $\mu = \mu_0$ H_1: $\mu < \mu_0$	$t_0 = \dfrac{\overline{X} - \mu_0}{s/\sqrt{n}}$	$t_0 < -t_{\alpha,\, n-1}$
H_0: $\mu = \mu_0$ H_1: $\mu > \mu_0$		$t_0 > t_{\alpha,\, n-1}$
H_0: $\mu_1 - \mu_2 = \gamma$ H_1: $\mu_1 - \mu_2 \neq \gamma$	Variances equal: $t_0 = \dfrac{\overline{X}_1 - \overline{X}_2 - \gamma}{s_p\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}}$, $v = n_1 + n_2 - 2$	$\lvert t_0 \rvert > t_{\alpha/2,\, v}$
H_0: $\mu_1 - \mu_2 = \gamma$ H_1: $\mu_1 - \mu_2 < \gamma$	Variances unequal: $t_0 = \dfrac{\overline{X}_1 - \overline{X}_2 - \gamma}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}}$	$t_0 < -t_{\alpha,\, v}$
H_0: $\mu_1 - \mu_2 = \gamma$ H_1: $\mu_1 - \mu_2 > \gamma$	$v = \dfrac{\left(\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}\right)^2}{\dfrac{\left(s_1^2/n_1\right)^2}{n_1 - 1} + \dfrac{\left(s_2^2/n_2\right)^2}{n_2 - 1}}$	$t_0 > t_{\alpha,\, v}$

In this table, $s_p^2 = [(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2]/v$

Tests on Variances of Normal Distribution with Unknown Means

Hypothesis	*Test Statistic*	*Criteria for Rejection*
H_0: $\sigma^2 = \sigma_0^2$ H_1: $\sigma^2 \neq \sigma_0^2$		$X_0^2 > X_{\alpha/2,n-1}^2$ *or* $X_0^2 < X_{1-\alpha/2,n-1}^2$
H_0: $\sigma^2 = \sigma_0^2$ H_1: $\sigma^2 < \sigma_0^2$	$X_0^2 = \dfrac{(n-1)s^2}{\sigma_0^2}$	$X_0^2 < X_{1-\alpha,n-1}^2$
H_0: $\sigma^2 = \sigma_0^2$ H_1: $\sigma^2 > \sigma_0^2$		$X_0^2 > X_{\alpha,n-1}^2$
H_0: $\sigma_1^2 = \sigma_2^2$ H_1: $\sigma_1^2 \neq \sigma_2^2$	$F_0 = \dfrac{s_1^2}{s_2^2}$	$F_0 > F_{\alpha/2,n_1-1,n_2-1}$ $F_0 < F_{1-\alpha/2,n_1-1,n_2-1}$
H_0: $\sigma_1^2 = \sigma_2^2$ H_1: $\sigma_1^2 < \sigma_2^2$	$F_0 = \dfrac{s_2^2}{s_1^2}$	$F_0 > F_{\alpha,n_2-1,n_1-1}$
H_0: $\sigma_1^2 = \sigma_2^2$ H_1: $\sigma_1^2 > \sigma_2^2$	$F_0 = \dfrac{s_1^2}{s_2^2}$	$F_0 > F_{\alpha,n_1-1,n_2-1}$

Paired t-Test

Null hypothesis: $H_0: \mu_D = \Delta_0$

Test statistic: $t_0 = \dfrac{\bar{D} - \Delta_0}{S_D / \sqrt{n}}$

Alternative Hypothesis	Rejection Region
$H_1: \mu_D \neq \Delta_0$	$t_0 > t_{\frac{\alpha}{2},n-1}$ or $t_0 < -t_{\frac{\alpha}{2},n-1}$
$H_1: \mu_D > \Delta_0$	$t_0 > t_{\alpha,n-1}$
$H_1: \mu_D < \Delta_0$	$t_0 < -t_{\alpha,n-1}$

where $\bar{D}$ is the sample average of the n differences $D_1, D_2, \ldots D_n$, and S_D is the sample standard deviation of these differences.

Sample Size

Sample Size Calculations for Population Proportions:

$$n = \frac{p(1-p)\left(Z_{\frac{\alpha}{2}}\right)^2}{e^2}$$

where

p = percentage occurrence of the element being sought, expressed as a decimal

n = total number of random observations

e = desired relative accuracy

Sample Size Calculations for Difference in Mean, Variances Known:

$$n = \frac{(\sigma_1^2 + \sigma_2^2)\left(Z_{\frac{\alpha}{2}}\right)^2}{E^2}$$

where

E = desired absolute accuracy

Sample Size Calculation for Confidence Interval, Variance Known:

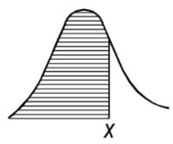
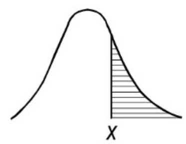
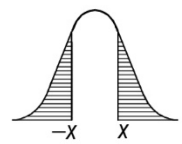
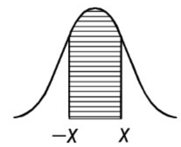
$$n = \left(\frac{\sigma Z_{\frac{\alpha}{2}}}{E}\right)^2$$

where

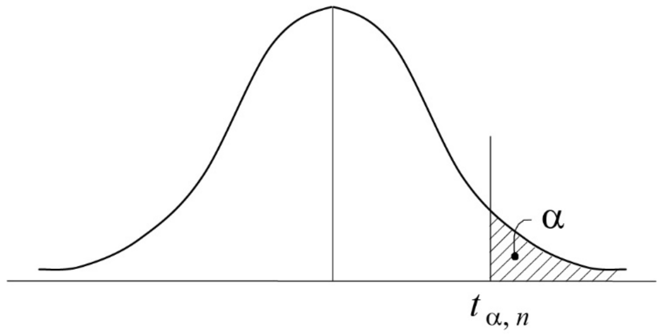
E = desired absolute accuracy

Unit Normal Distribution Table

$(\mu = 0, \ \sigma = 1)$

					
x	$f(x)$	$F(x)$	$R(x)$	$2R(x)$	$W(x)$
0.0	0.3989	0.5000	0.5000	1.0000	0.0000
0.1	0.3970	0.5398	0.4602	0.9203	0.0797
0.2	0.3910	0.5793	0.4207	0.8415	0.1585
0.3	0.3814	0.6179	0.3821	0.7642	0.2358
0.4	0.3683	0.6554	0.3446	0.6892	0.3108
0.5	0.3521	0.6915	0.3085	0.6171	0.3829
0.6	0.3332	0.7257	0.2743	0.5485	0.4515
0.7	0.3123	0.7580	0.2420	0.4839	0.5161
0.8	0.2897	0.7881	0.2119	0.4237	0.5763
0.9	0.2661	0.8159	0.1841	0.3681	0.6319
1.0	0.2420	0.8413	0.1 587	0.3173	0.6827
1.1	0.2179	0.8643	0.1357	0.2713	0.7287
1.2	0.1942	0.8849	0.1151	0.2301	0.7699
1.3	0.1714	0.9032	0.0968	0.1936	0.8064
1.4	0.1497	0.9192	0.0808	0.1615	0.8385
1.5	0.1295	0.9332	0.0668	0.1336	0.8664
1.6	0.1109	0.9452	0.0548	0.1096	0.8904
1.7	0.0940	0.9554	0.0446	0.0891	0.9109
1.8	0.0790	0.9641	0.0359	0.0719	0.9281
1.9	0.0656	0.9713	0.0287	0.0574	0.9426
2.0	0.0540	0.9772	0.0228	0.0455	0.9545
2.1	0.044 0	0.9821	0.0179	0.0357	0.9643
2.2	0.0355	0.9861	0.0139	0.0278	0.9722
2.3	0.0283	0.9893	0.0107	0.0214	0.9786
2.4	0.0224	0.9918	0.0082	0.0164	0.9836
2.5	0.0175	0.9938	0.0062	0.0124	0.9876
2.6	0.0136	0.9953	0.0047	0.0093	0.9907
2.7	0.0104	0.9965	0.0035	0.0069	0.9931
2.8	0.0079	0.9974	0.0026	0.0051	0.9949
2.9	0.0060	0.9981	0.0019	0.0037	0.9963
3.0	0.0044	0.9987	0.0013	0.0027	0.99 73
Fractiles					
1.2816	0.1755	0.9000	0.1000	0.2000	0.8000
1.6449	0.1031	0.9500	0.0500	0.1000	0.9000
1.9600	0.0584	0.9750	0.0250	0.0500	0.9500
2.0537	0.0484	0.9800	0.0200	0.0400	0.9600
2.3263	0.0267	0.9900	0.0100	0.0200	0.9800
2.5758	0.0145	0.9950	0.0050	0.0100	0.9900

t-Distribution Table



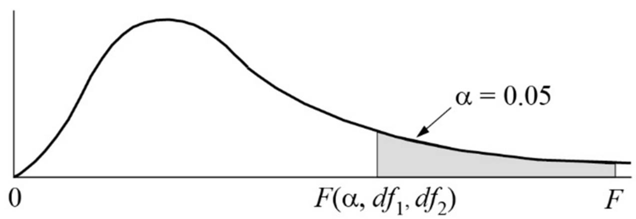
Values of $t_{\alpha,n}$

n	α = 0.10	α = 0.05	α = 0.025	α = 0.01	α = 0.005	*n*
1	3.078	6.314	12.706	31.821	63.657	1
2	1.886	2.920	4.303	6.965	9.925	2
3	1.638	2.353	3.182	4.541	5.841	3
4	1.533	2.132	2.776	3.747	4.604	4
5	1.476	2.015	2.571	3.365	4.032	5
6	1.440	1.943	2.447	3.143	3.707	6
7	1.415	1.895	2.365	2.998	3.499	7
8	1.397	1.860	2.306	2.896	3.355	8
9	1.383	1.833	2.262	2.821	3.250	9
10	1.372	1.812	2.228	2.764	3.169	10
11	1.363	1.796	2.201	2.718	3.106	11
12	1.356	1.782	2.179	2.681	3.055	12
13	1.350	1.771	2.160	2.650	3.012	13
14	1.345	1.761	2.145	2.624	2.977	14
15	1.341	1.753	2.131	2.602	2.947	15
16	1.337	1.746	2.120	2.583	2.921	16
17	1.333	1.740	2.110	2.567	2.898	17
18	1.330	1.734	2.101	2.552	2.878	18
19	1.328	1.729	2.093	2.539	2.861	19
20	1.325	1.725	2.086	2.528	2.845	20
21	1.323	1.721	2.080	2.518	2.831	21
22	1.321	1.717	2.074	2.508	2.819	22
23	1.319	1.714	2.069	2.500	2.807	23
24	1.318	1.711	2.064	2.492	2.797	24
25	1.316	1.708	2.060	2.485	2.787	25
26	1.315	1.706	2.056	2.479	2.779	26
27	1.314	1.703	2.052	2.473	2.771	27
28	1.313	1.701	2.048	2.467	2.763	28
29	1.311	1.699	2.045	2.462	2.756	29
∞	1.282	1.645	1.960	2.326	2.576	∞

F-Distribution

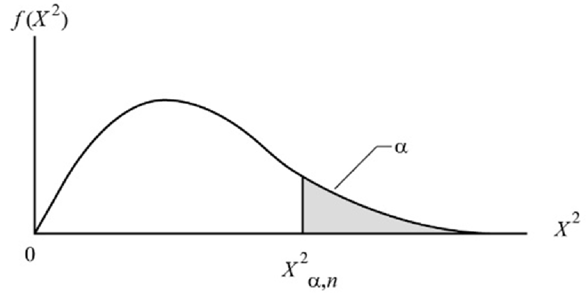
CRITICAL VALUES OF THE *F* DISTRIBUTION – TABLE

For a particular combination of numerator and denominator degrees of freedom, entry represents the critical values of *F* corresponding to a specified upper tail area (α).



Denominator df_2	Numerator df_1: 1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
1	161.4	199.5	215.7	224.6	230.2	234.0	236.8	238.9	240.5	241.9	243.9	245.9	248.0	249.1	250.1	251.1	252.2	253.3	254.3
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.41	19.43	19.45	19.45	19.46	19.47	19.48	19.49	19.50
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57	8.55	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
5	**6.61**	**5.79**	**5.41**	**5.19**	**5.05**	**4.95**	**4.88**	**4.82**	**4.77**	**4.74**	**4.68**	**4.62**	**4.56**	**4.53**	**4.50**	**4.46**	**4.43**	**4.40**	**4.36**
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74	3.70	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30	3.27	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01	2.97	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79	2.75	2.71
10	**4.96**	**4.10**	**3.71**	**3.48**	**3.33**	**3.22**	**3.14**	**3.07**	**3.02**	**2.98**	**2.91**	**2.85**	**2.77**	**2.74**	**2.70**	**2.66**	**2.62**	**2.58**	**2.54**
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49	2.45	2.40
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38	2.34	2.30
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.53	2.46	2.42	2.38	2.34	2.30	2.25	2.21
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22	2.18	2.13
15	**4.54**	**3.68**	**3.29**	**3.06**	**2.90**	**2.79**	**2.71**	**2.64**	**2.59**	**2.54**	**2.48**	**2.40**	**2.33**	**2.29**	**2.25**	**2.20**	**2.16**	**2.11**	**2.07**
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11	2.06	2.01
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06	2.01	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02	1.97	1.92
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
20	**4.35**	**3.49**	**3.10**	**2.87**	**2.71**	**2.60**	**2.51**	**2.45**	**2.39**	**2.35**	**2.28**	**2.20**	**2.12**	**2.08**	**2.04**	**1.99**	**1.95**	**1.90**	**1.84**
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92	1.87	1.81
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.15	2.07	2.03	1.98	1.94	1.89	1.84	1.78
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.13	2.05	2.01	1.96	1.91	1.86	1.81	1.76
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11	2.03	1.98	1.94	1.89	1.84	1.79	1.73
25	**4.24**	**3.39**	**2.99**	**2.76**	**2.60**	**2.49**	**2.40**	**2.34**	**2.28**	**2.24**	**2.16**	**2.09**	**2.01**	**1.96**	**1.92**	**1.87**	**1.82**	**1.77**	**1.71**
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.15	2.07	1.99	1.95	1.90	1.85	1.80	1.75	1.69
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20	2.13	2.06	1.97	1.93	1.88	1.84	1.79	1.73	1.67
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.12	2.04	1.96	1.91	1.87	1.82	1.77	1.71	1.65
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18	2.10	2.03	1.94	1.90	1.85	1.81	1.75	1.70	1.64
30	**4.17**	**3.32**	**2.92**	**2.69**	**2.53**	**2.42**	**2.33**	**2.27**	**2.21**	**2.16**	**2.09**	**2.01**	**1.93**	**1.89**	**1.84**	**1.79**	**1.74**	**1.68**	**1.62**
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.92	1.84	1.79	1.74	1.69	1.64	1.58	1.51
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.84	1.75	1.70	1.65	1.59	1.53	1.47	1.39
120	3.92	3.07	2.68	2.45	2.29	2.17	2.09	2.02	1.96	1.91	1.83	1.75	1.66	1.61	1.55	1.50	1.43	1.35	1.25
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.75	1.67	1.57	1.52	1.46	1.39	1.32	1.22	1.00

CRITICAL VALUES OF X^2 DISTRIBUTION



Degrees of Freedom	$X^2_{.995}$	$X^2_{.990}$	$X^2_{.975}$	$X^2_{.950}$	$X^2_{.900}$	$X^2_{.100}$	$X^2_{.050}$	$X^2_{.025}$	$X^2_{.010}$	$X^2_{.005}$
1	0.0000393	0.0001571	0.0009821	0.0039321	0.0157908	2.70554	3.84146	5.02389	6.63490	7.87944
2	0.0100251	0.0201007	0.0506356	0.102587	0.210720	4.60517	5.99147	7.37776	9.21034	10.5966
3	0.0717212	0.114832	0.215795	0.351846	0.584375	6.25139	7.81473	9.34840	11.3449	12.8381
4	0.206990	0.297110	0.484419	0.710721	1.063623	7.77944	9.48773	11.1433	13.2767	14.8602
5	0.411740	0.554300	0.831211	1.145476	1.61031	9.23635	11.0705	12.8325	15.0863	16.7496
6	0.675727	0.872085	1.237347	1.63539	2.20413	10.6446	12.5916	14.4494	16.8119	18.5476
7	0.989265	1.239043	1.68987	2.16735	2.83311	12.0170	14.0671	16.0128	18.4753	20.2777
8	1.344419	1.646482	2.17973	2.73264	3.48954	13.3616	15.5073	17.5346	20.0902	21.9550
9	1.734926	2.087912	2.70039	3.32511	4.16816	14.6837	16.9190	19.0228	21.6660	23.5893
10	2.15585	2.55821	3.24697	3.94030	4.86518	15.9871	18.3070	20.4831	23.2093	25.1882
11	2.60321	3.05347	3.81575	4.57481	5.57779	17.2750	19.6751	21.9200	24.7250	26.7569
12	3.07382	3.57056	4.40379	5.22603	6.30380	18.5494	21.0261	23.3367	26.2170	28.2995
13	3.56503	4.10691	5.00874	5.89186	7.04150	19.8119	22.3621	24.7356	27.6883	29.8194
14	4.07468	4.66043	5.62872	6.57063	7.78953	21.0642	23.6848	26.1190	29.1413	31.3193
15	4.60094	5.22935	6.26214	7.26094	8.54675	22.3072	24.9958	27.4884	30.5779	32.8013
16	5.14224	5.81221	6.90766	7.96164	9.31223	23.5418	26.2962	28.8454	31.9999	34.2672
17	5.69724	6.40776	7.56418	8.67176	10.0852	24.7690	27.5871	30.1910	33.4087	35.7185
18	6.26481	7.01491	8.23075	9.39046	10.8649	25.9894	28.8693	31.5264	34.8053	37.1564
19	6.84398	7.63273	8.90655	10.1170	11.6509	27.2036	30.1435	32.8523	36.1908	38.5822
20	7.43386	8.26040	9.59083	10.8508	12.4426	28.4120	31.4104	34.1696	37.5662	39.9968
21	8.03366	8.89720	10.28293	11.5913	13.2396	29.6151	32.6705	35.4789	38.9321	41.4010
22	8.64272	9.54249	10.9823	12.3380	14.0415	30.8133	33.9244	36.7807	40.2894	42.7956
23	9.26042	10.19567	11.6885	13.0905	14.8479	32.0069	35.1725	38.0757	41.6384	44.1813
24	9.88623	10.8564	12.4011	13.8484	15.6587	33.1963	36.4151	39.3641	42.9798	45.5585
25	10.5197	11.5240	13.1197	14.6114	16.4734	34.3816	37.6525	40.6465	44.3141	46.9278
26	11.1603	12.1981	13.8439	15.3791	17.2919	35.5631	38.8852	41.9232	45.6417	48.2899
27	11.8076	12.8786	14.5733	16.1513	18.1138	36.7412	40.1133	43.1944	46.9630	49.6449
28	12.4613	13.5648	15.3079	16.9279	18.9392	37.9159	41.3372	44.4607	48.2782	50.9933
29	13.1211	14.2565	16.0471	17.7083	19.7677	39.0875	42.5569	45.7222	49.5879	52.3356
30	13.7867	14.9535	16.7908	18.4926	20.5992	40.2560	43.7729	46.9792	50.8922	53.6720
40	20.7065	22.1643	24.4331	26.5093	29.0505	51.8050	55.7585	59.3417	63.6907	66.7659
50	27.9907	29.7067	32.3574	34.7642	37.6886	63.1671	67.5048	71.4202	76.1539	79.4900
60	35.5346	37.4848	40.4817	43.1879	46.4589	74.3970	79.0819	83.2976	88.3794	91.9517
70	43.2752	45.4418	48.7576	51.7393	55.3290	85.5271	90.5312	95.0231	100.425	104.215
80	51.1720	53.5400	57.1532	60.3915	64.2778	96.5782	101.879	106.629	112.329	116.321
90	59.1963	61.7541	65.6466	69.1260	73.2912	107.565	113.145	118.136	124.116	128.299
100	67.3276	70.0648	74.2219	77.9295	82.3581	118.498	124.342	129.561	135.807	140.169

Source: Thompson, C. M., "Tables of the Percentage Points of the X^2–Distribution," *Biometrika*, ©1941, 32, 188-189. Reproduced by permission of Oxford University Press.

Cumulative Binomial Probabilities $P(X \le x)$

		P										
n	x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95	0.99
1	0	0.9000	0.8000	0.7000	0.6000	0.5000	0.4000	0.3000	0.2000	0.1000	0.0500	0.0100
2	0	0.8100	0.6400	0.4900	0.3600	0.2500	0.1600	0.0900	0.0400	0.0100	0.0025	0.0001
	1	0.9900	0.9600	0.9100	0.8400	0.7500	0.6400	0.5100	0.3600	0.1900	0.0975	0.0199
3	0	0.7290	0.5120	0.3430	0.2160	0.1250	0.0640	0.0270	0.0080	0.0010	0.0001	0.0000
	1	0.9720	0.8960	0.7840	0.6480	0.5000	0.3520	0.2160	0.1040	0.0280	0.0073	0.0003
	2	0.9990	0.9920	0.9730	0.9360	0.8750	0.7840	0.6570	0.4880	0.2710	0.1426	0.0297
4	0	0.6561	0.4096	0.2401	0.1296	0.0625	0.0256	0.0081	0.0016	0.0001	0.0000	0.0000
	1	0.9477	0.8192	0.6517	0.4752	0.3125	0.1792	0.0837	0.0272	0.0037	0.0005	0.0000
	2	0.9963	0.9728	0.9163	0.8208	0.6875	0.5248	0.3483	0.1808	0.0523	0.0140	0.0006
	3	0.9999	0.9984	0.9919	0.9744	0.9375	0.8704	0.7599	0.5904	0.3439	0.1855	0.0394
5	0	0.5905	0.3277	0.1681	0.0778	0.0313	0.0102	0.0024	0.0003	0.0000	0.0000	0.0000
	1	0.9185	0.7373	0.5282	0.3370	0.1875	0.0870	0.0308	0.0067	0.0005	0.0000	0.0000
	2	0.9914	0.9421	0.8369	0.6826	0.5000	0.3174	0.1631	0.0579	0.0086	0.0012	0.0000
	3	0.9995	0.9933	0.9692	0.9130	0.8125	0.6630	0.4718	0.2627	0.0815	0.0226	0.0010
	4	1.0000	0.9997	0.9976	0.9898	0.9688	0.9222	0.8319	0.6723	0.4095	0.2262	0.0490
6	0	0.5314	0.2621	0.1176	0.0467	0.0156	0.0041	0.0007	0.0001	0.0000	0.0000	0.0000
	1	0.8857	0.6554	0.4202	0.2333	0.1094	0.0410	0.0109	0.0016	0.0001	0.0000	0.0000
	2	0.9842	0.9011	0.7443	0.5443	0.3438	0.1792	0.0705	0.0170	0.0013	0.0001	0.0000
	3	0.9987	0.9830	0.9295	0.8208	0.6563	0.4557	0.2557	0.0989	0.0159	0.0022	0.0000
	4	0.9999	0.9984	0.9891	0.9590	0.8906	0.7667	0.5798	0.3446	0.1143	0.0328	0.0015
	5	1.0000	0.9999	0.9993	0.9959	0.9844	0.9533	0.8824	0.7379	0.4686	0.2649	0.0585
7	0	0.4783	0.2097	0.0824	0.0280	0.0078	0.0016	0.0002	0.0000	0.0000	0.0000	0.0000
	1	0.8503	0.5767	0.3294	0.1586	0.0625	0.0188	0.0038	0.0004	0.0000	0.0000	0.0000
	2	0.9743	0.8520	0.6471	0.4199	0.2266	0.0963	0.0288	0.0047	0.0002	0.0000	0.0000
	3	0.9973	0.9667	0.8740	0.7102	0.5000	0.2898	0.1260	0.0333	0.0027	0.0002	0.0000
	4	0.9998	0.9953	0.9712	0.9037	0.7734	0.5801	0.3529	0.1480	0.0257	0.0038	0.0000
	5	1.0000	0.9996	0.9962	0.9812	0.9375	0.8414	0.6706	0.4233	0.1497	0.0444	0.0020
	6	1.0000	1.0000	0.9998	0.9984	0.9922	0.9720	0.9176	0.7903	0.5217	0.3017	0.0679
8	0	0.4305	0.1678	0.0576	0.0168	0.0039	0.0007	0.0001	0.0000	0.0000	0.0000	0.0000
	1	0.8131	0.5033	0.2553	0.1064	0.0352	0.0085	0.0013	0.0001	0.0000	0.0000	0.0000
	2	0.9619	0.7969	0.5518	0.3154	0.1445	0.0498	0.0113	0.0012	0.0000	0.0000	0.0000
	3	0.9950	0.9437	0.8059	0.5941	0.3633	0.1737	0.0580	0.0104	0.0004	0.0000	0.0000
	4	0.9996	0.9896	0.9420	0.8263	0.6367	0.4059	0.1941	0.0563	0.0050	0.0004	0.0000
	5	1.0000	0.9988	0.9887	0.9502	0.8555	0.6846	0.4482	0.2031	0.0381	0.0058	0.0001
	6	1.0000	0.9999	0.9987	0.9915	0.9648	0.8936	0.7447	0.4967	0.1869	0.0572	0.0027
	7	1.0000	1.0000	0.9999	0.9993	0.9961	0.9832	0.9424	0.8322	0.5695	0.3366	0.0773
9	0	0.3874	0.1342	0.0404	0.0101	0.0020	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.7748	0.4362	0.1960	0.0705	0.0195	0.0038	0.0004	0.0000	0.0000	0.0000	0.0000
	2	0.9470	0.7382	0.4628	0.2318	0.0898	0.0250	0.0043	0.0003	0.0000	0.0000	0.0000
	3	0.9917	0.9144	0.7297	0.4826	0.2539	0.0994	0.0253	0.0031	0.0001	0.0000	0.0000
	4	0.9991	0.9804	0.9012	0.7334	0.5000	0.2666	0.0988	0.0196	0.0009	0.0000	0.0000
	5	0.9999	0.9969	0.9747	0.9006	0.7461	0.5174	0.2703	0.0856	0.0083	0.0006	0.0000
	6	1.0000	0.9997	0.9957	0.9750	0.9102	0.7682	0.5372	0.2618	0.0530	0.0084	0.0001
	7	1.0000	1.0000	0.9996	0.9962	0.9805	0.9295	0.8040	0.5638	0.2252	0.0712	0.0034
	8	1.0000	1.0000	1.0000	0.9997	0.9980	0.9899	0.9596	0.8658	0.6126	0.3698	0.0865

Montgomery, Douglas C., and George C. Runger, *Applied Statistics and Probability for Engineers*, 4 ed., New York: John Wiley and Sons, 2007.

Cumulative Binomial Probabilities $P(X \leq x)$ (*continued*)

		P										
n	x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95	0.99
10	0	0.3487	0.1074	0.0282	0.0060	0.0010	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.7361	0.3758	0.1493	0.0464	0.0107	0.0017	0.0001	0.0000	0.0000	0.0000	0.0000
	2	0.9298	0.6778	0.3828	0.1673	0.0547	0.0123	0.0016	0.0001	0.0000	0.0000	0.0000
	3	0.9872	0.8791	0.6496	0.3823	0.1719	0.0548	0.0106	0.0009	0.0000	0.0000	0.0000
	4	0.9984	0.9672	0.8497	0.6331	0.3770	0.1662	0.0473	0.0064	0.0001	0.0000	0.0000
	5	0.9999	0.9936	0.9527	0.8338	0.6230	0.3669	0.1503	0.0328	0.0016	0.0001	0.0000
	6	1.0000	0.9991	0.9894	0.9452	0.8281	0.6177	0.3504	0.1209	0.0128	0.0010	0.0000
	7	1.0000	0.9999	0.9984	0.9877	0.9453	0.8327	0.6172	0.3222	0.0702	0.0115	0.0001
	8	1.0000	1.0000	0.9999	0.9983	0.9893	0.9536	0.8507	0.6242	0.2639	0.0861	0.0043
	9	1.0000	1.0000	1.0000	0.9999	0.9990	0.9940	0.9718	0.8926	0.6513	0.4013	0.0956
15	0	0.2059	0.0352	0.0047	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.5490	0.1671	0.0353	0.0052	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	2	0.8159	0.3980	0.1268	0.0271	0.0037	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000
	3	0.9444	0.6482	0.2969	0.0905	0.0176	0.0019	0.0001	0.0000	0.0000	0.0000	0.0000
	4	0.9873	0.8358	0.5155	0.2173	0.0592	0.0093	0.0007	0.0000	0.0000	0.0000	0.0000
	5	0.9978	0.9389	0.7216	0.4032	0.1509	0.0338	0.0037	0.0001	0.0000	0.0000	0.0000
	6	0.9997	0.9819	0.8689	0.6098	0.3036	0.0950	0.0152	0.0008	0.0000	0.0000	0.0000
	7	1.0000	0.9958	0.9500	0.7869	0.5000	0.2131	0.0500	0.0042	0.0000	0.0000	0.0000
	8	1.0000	0.9992	0.9848	0.9050	0.6964	0.3902	0.1311	0.0181	0.0003	0.0000	0.0000
	9	1.0000	0.9999	0.9963	0.9662	0.8491	0.5968	0.2784	0.0611	0.0022	0.0001	0.0000
	10	1.0000	1.0000	0.9993	0.9907	0.9408	0.7827	0.4845	0.1642	0.0127	0.0006	0.0000
	11	1.0000	1.0000	0.9999	0.9981	0.9824	0.9095	0.7031	0.3518	0.0556	0.0055	0.0000
	12	1.0000	1.0000	1.0000	0.9997	0.9963	0.9729	0.8732	0.6020	0.1841	0.0362	0.0004
	13	1.0000	1.0000	1.0000	1.0000	0.9995	0.9948	0.9647	0.8329	0.4510	0.1710	0.0096
	14	1.0000	1.0000	1.0000	1.0000	1.0000	0.9995	0.9953	0.9648	0.7941	0.5367	0.1399
20	0	0.1216	0.0115	0.0008	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.3917	0.0692	0.0076	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	2	0.6769	0.2061	0.0355	0.0036	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	3	0.8670	0.4114	0.1071	0.0160	0.0013	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	4	0.9568	0.6296	0.2375	0.0510	0.0059	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000
	5	0.9887	0.8042	0.4164	0.1256	0.0207	0.0016	0.0000	0.0000	0.0000	0.0000	0.0000
	6	0.9976	0.9133	0.6080	0.2500	0.0577	0.0065	0.0003	0.0000	0.0000	0.0000	0.0000
	7	0.9996	0.9679	0.7723	0.4159	0.1316	0.0210	0.0013	0.0000	0.0000	0.0000	0.0000
	8	0.9999	0.9900	0.8867	0.5956	0.2517	0.0565	0.0051	0.0001	0.0000	0.0000	0.0000
	9	1.0000	0.9974	0.9520	0.7553	0.4119	0.1275	0.0171	0.0006	0.0000	0.0000	0.0000
	10	1.0000	0.9994	0.9829	0.8725	0.5881	0.2447	0.0480	0.0026	0.0000	0.0000	0.0000
	11	1.0000	0.9999	0.9949	0.9435	0.7483	0.4044	0.1133	0.0100	0.0001	0.0000	0.0000
	12	1.0000	1.0000	0.9987	0.9790	0.8684	0.5841	0.2277	0.0321	0.0004	0.0000	0.0000
	13	1.0000	1.0000	0.9997	0.9935	0.9423	0.7500	0.3920	0.0867	0.0024	0.0000	0.0000
	14	1.0000	1.0000	1.0000	0.9984	0.9793	0.8744	0.5836	0.1958	0.0113	0.0003	0.0000
	15	1.0000	1.0000	1.0000	0.9997	0.9941	0.9490	0.7625	0.3704	0.0432	0.0026	0.0000
	16	1.0000	1.0000	1.0000	1.0000	0.9987	0.9840	0.8929	0.5886	0.1330	0.0159	0.0000
	17	1.0000	1.0000	1.0000	1.0000	0.9998	0.9964	0.9645	0.7939	0.3231	0.0755	0.0010
	18	1.0000	1.0000	1.0000	1.0000	1.0000	0.9995	0.9924	0.9308	0.6083	0.2642	0.0169
	19	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9992	0.9885	0.8784	0.6415	0.1821

Montgomery, Douglas C., and George C. Runger, *Applied Statistics and Probability for Engineers*, 4 ed., New York: John Wiley and Sons, 2007.

Linear Regression

Least Squares

$\hat{y} = \hat{a} + \hat{b}x$, where

$\hat{b} = S_{xy}/S_{xx}$

$\hat{a} = \overline{y} - \hat{b}\overline{x}$

$$S_{xy} = \sum_{i=1}^{n} x_i y_i - (1/n)\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)$$

$$S_{xx} = \sum_{i=1}^{n} x_i^2 - (1/n)\left(\sum_{i=1}^{n} x_i\right)^2$$

$$\overline{y} = (1/n)\left(\sum_{i=1}^{n} y_i\right)$$

$$\overline{x} = (1/n)\left(\sum_{i=1}^{n} x_i\right)$$

where

n = sample size
S_{xx} = sum of squares of x
S_{yy} = sum of squares of y
S_{xy} = sum of x-y products

Residual

$$e_i = y_i - \hat{y} = y_i - \left(\hat{a} + \hat{b}x_i\right)$$

Standard Error of Estimate

$S_e^2 = \frac{S_{xx}S_{yy} - S_{xy}^2}{S_{xx}(n-2)} = MSE$, where

$$S_{yy} = \sum_{i=1}^{n} y_i^2 - (1/n)\left(\sum_{i=1}^{n} y_i\right)^2$$

Confidence Interval for â

$$\hat{a} \pm t_{\alpha/2,n-2}\sqrt{\left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}\right)MSE}$$

Confidence Interval for $\hat{b}$

$$\hat{b} \pm t_{\alpha/2,n-2}\sqrt{\frac{MSE}{S_{xx}}}$$

Sample Correlation Coefficient

$$r = \frac{S_{xy}}{\sqrt{S_{xx}S_{yy}}}$$

Mathematics

Straight Line

The general form of the equation is $Ax + By + C = 0$

The standard form of the equation is $y = mx + b$, which is also known as the *slope-intercept* form.

The *point-slope* form is $y - y_1 = m(x - x_1)$

Given two points: slope, $m = (y_2 - y_1)/(x_2 - x_1)$

The angle between lines with slopes m_1 and m_2 is $\alpha = \arctan\,[(m_2 - m_1)/(1 + m_2 \cdot m_1)]$

Two lines are perpendicular if $m_1 = -1/m_2$

The distance between two points is $d = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$

Quadratic Equations

$ax^2 + bx + c = 0$

$$Roots = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Logarithms

The logarithm of x to the Base b is defined by

$\log_b (x) = c$, where $b^c = x$

Special definitions for $b = e$ or $b = 10$ are:

$\ln x$, Base = e

$\log x$, Base = 10

To change from one Base to another:

$\log_b x = (\log_a x)/(\log_a b)$

e.g., $\ln x = (\log_{10} x)/(\log_{10} e) = 2.302585\ (\log_{10} x)$

Identities

$\log_b b^n = n$

$\log x^c = c \log x$; $x^c = \text{antilog}\,(c \log x)$

$\log xy = \log x + \log y$

$\log_b b = 1$; $\log 1 = 0$

$\log x/y = \log x - \log y$

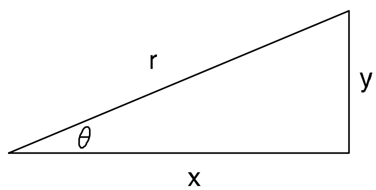
Trigonometry

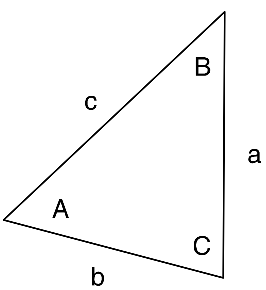
Trigonometric functions are defined using a right triangle.

$\sin \theta = y/r$, $\cos \theta = x/r$

$\tan \theta = y/x$, $\cot \theta = x/y$

$\csc \theta = r/y$, $\sec \theta = r/x$





Law of Sines $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Law of Cosines

$a^2 = b^2 + c^2 - 2bc \cos A$

$b^2 = a^2 + c^2 - 2ac \cos B$

$c^2 = a^2 + b^2 - 2ab \cos C$

Brink, R.W., *A First Year of College Mathematics*, Copyright © 1937 by D. Appleton-Century Co., Inc. Prentice-Hall, Inc., Englewood Cliffs, NJ.

Identities

$\csc\theta = 1/\sin\theta$

$\sec\theta = 1/\cos\theta$

$\tan\theta = \sin\theta/\cos\theta$

$\cot\theta = 1/\tan\theta$

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$\cot^2\theta + 1 = \csc^2\theta$

$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\sin 2\alpha = 2\sin\alpha\cos\alpha$

$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha = 2\cos^2\alpha - 1$

$\tan 2\alpha = (2\tan\alpha)/(1 - \tan^2\alpha)$

$\cot 2\alpha = (\cot^2\alpha - 1)/(2\cot\alpha)$

$\tan(\alpha + \beta) = (\tan\alpha + \tan\beta)/(1 - \tan\alpha\tan\beta)$

$\cot(\alpha + \beta) = (\cot\alpha\cot\beta - 1)/(\cot\alpha + \cot\beta)$

$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$\tan(\alpha - \beta) = (\tan\alpha - \tan\beta)/(1 + \tan\alpha\tan\beta)$

$\cot(\alpha - \beta) = (\cot\alpha\cot\beta + 1)/(\cot\beta - \cot\alpha)$

$\sin(\alpha/2) = \pm\sqrt{(1 - \cos\alpha)/2}$

$\cos(\alpha/2) = \pm\sqrt{(1 + \cos\alpha)/2}$

$\tan(\alpha/2) = \pm\sqrt{(1 - \cos\alpha)/(1 + \cos\alpha)}$

$\cot(\alpha/2) = \pm\sqrt{(1 + \cos\alpha)/(1 - \cos\alpha)}$

$\sin\alpha\sin\beta = (1/2)[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$

$\cos\alpha\cos\beta = (1/2)[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$

$\sin\alpha\cos\beta = (1/2)[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$

$\sin\alpha + \sin\beta = 2\sin(1/2)(\alpha + \beta)\cos(1/2)(\alpha - \beta)$

$\sin\alpha - \sin\beta = 2\cos(1/2)(\alpha + \beta)\sin(1/2)(\alpha - \beta)$

$\cos\alpha + \cos\beta = 2\cos(1/2)(\alpha + \beta)\cos(1/2)(\alpha - \beta)$

$\cos\alpha - \cos\beta = -2\sin(1/2)(\alpha + \beta)\sin(1/2)(\alpha - \beta)$

Complex Numbers

Definition $i = \sqrt{-1}$

$(a + ib) + (c + id) = (a + c) + i\,(b + d)$

$(a + ib) - (c + id) = (a - c) + i\,(b - d)$

$(a + ib)(c + id) = (ac - bd) + i\,(ad + bc)$

$$\frac{a+ib}{c+id} = \frac{(a+ib)(c-id)}{(c+id)(c-id)} = \frac{(ac+bd)+i(bc-ad)}{c^2+d^2}$$

$(a + ib) + (a - ib) = 2a$

$(a + ib) - (a - ib) = 2ib$

$(a + ib)(a - ib) = a^2 + b^2$

Matrices

A matrix is an ordered rectangular array of numbers with m rows and n columns. The element a_{ij} refers to row i and column j.

Multiplication

If $\boldsymbol{A} = (a_{ik})$ is an $m \times n$ matrix and $\boldsymbol{B} = (b_{kj})$ is an $n \times s$ matrix, the matrix product AB is an $m \times s$ matrix

$$C = \left(c_{i\,j}\right) = \left(\sum_{l=1}^{n} a_{il} b_{lj}\right)$$

where n is the common integer representing the number of columns of $\boldsymbol{A}$ and the number of rows of $\boldsymbol{B}$ (l and k = 1, 2, …, n).

Addition

If $\boldsymbol{A} = (a_{ij})$ and $\boldsymbol{B} = (b_{ij})$ are two matrices of the same size $m \times n$, the sum $\boldsymbol{A} + \boldsymbol{B}$ is the $m \times n$ matrix $\boldsymbol{C} = (c_{ij})$

where $c_{ij} = a_{ij} + b_{ij}$.

Identity

The matrix $\mathbf{I} = (a_{ij})$ is a square $n \times n$ identity matrix where $a_{ii} = 1$ for $i = 1, 2, \ldots, n$ and $a_{ij} = 0$ for $i \neq j$.

Transpose

The matrix $\boldsymbol{B}$ is the transpose of the matrix $\boldsymbol{A}$ if each entry b_{ji} in $\boldsymbol{B}$ is the same as the entry a_{ij} in $\boldsymbol{A}$ and conversely. In equation form, the transpose is $\boldsymbol{B} = \boldsymbol{A}^T$.

Inverse

The inverse $\boldsymbol{B}$ of a square $n \times n$ matrix $\boldsymbol{A}$ is

$$\boldsymbol{B} = \boldsymbol{A}^{-1} = \frac{adj(\boldsymbol{A})}{|\boldsymbol{A}|}$$

where adj($\boldsymbol{A}$) = adjoint of $\boldsymbol{A}$ (obtained by replacing $\boldsymbol{A}^T$ elements with their cofactors, see **DETERMINANTS**) and

$|\boldsymbol{A}|$ = determinant of $\boldsymbol{A}$.

Determinants

A *determinant of order n* consists of n^2 numbers, called the *elements* of the determinant, arranged in *n* rows and *n* columns and enclosed by two vertical lines. In any determinant, the *minor* of a given element is the determinant that remains after all of the elements are struck out that lie in the same row and in the same column as the given element. Consider an element which lies in the *h*th column and the *k*th row. The *cofactor* of this element is the value of the minor of the element (if $h + k$ is *even*), and it is the negative of the value of the minor of the element (if $h + k$ is *odd*).

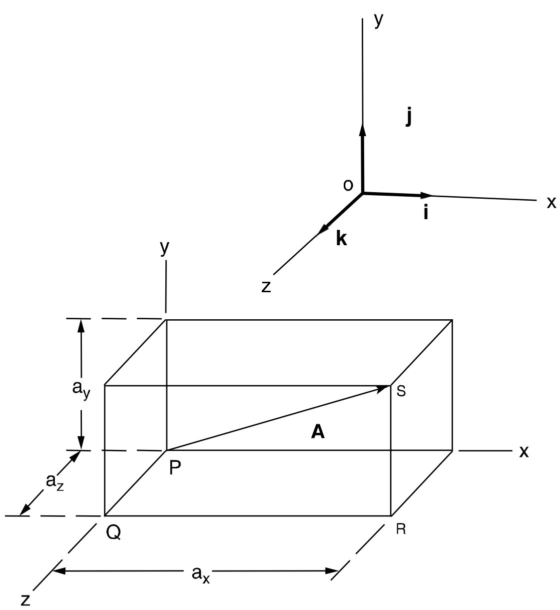
If *n* is greater than 1, the *value* of a determinant of order *n* is the sum of the *n* products formed by multiplying each element of some specified row (or column) by its cofactor. This sum is called the *expansion of the determinant* [according to the elements of the specified row (or column)]. For a second-order determinant:

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

For a third-order determinant:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1 - a_2 b_1 c_3 - a_1 b_3 c_2$$

Vectors



$\mathbf{A} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$

Addition and *subtraction:*

$\mathbf{A} + \mathbf{B} = (a_x + b_x)\mathbf{i} + (a_y + b_y)\mathbf{j} + (a_z + b_z)\mathbf{k}$

$\mathbf{A} - \mathbf{B} = (a_x - b_x)\mathbf{i} + (a_y - b_y)\mathbf{j} + (a_z - b_z)\mathbf{k}$

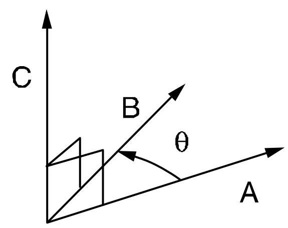
The *dot product* is a *scalar product* and represents the projection of **B** onto **A** times $|\mathbf{A}|$. It is given by

$$\mathbf{A}\bullet\mathbf{B} = a_x b_x + a_y b_y + a_z b_z$$

$$= |\mathbf{A}|\,|\mathbf{B}| \cos\theta = \mathbf{B}\bullet\mathbf{A}$$

The *cross product* is a *vector product* of magnitude $|\mathbf{B}|\,|\mathbf{A}| \sin\theta$ which is perpendicular to the plane containing **A** and **B**. The product is

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = -\mathbf{B}\times\mathbf{A}$$



The sense of $\mathbf{A}\times\mathbf{B}$ is determined by the right-hand rule.

$$\mathbf{A}\times\mathbf{B} = |\mathbf{A}|\,|\mathbf{B}|\,\mathbf{n}\sin\theta, \text{ where}$$

n = unit vector perpendicular to the plane of **A** and **B**.

Gradient, Divergence, and Curl

$$\nabla\varphi = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\varphi$$

$$\nabla\bullet\mathbf{V} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\bullet\left(V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}\right)$$

$$\nabla\times\mathbf{V} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\times\left(V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}\right)$$

The Laplacian of a scalar function ϕ is

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

Identities

$\mathbf{A} \bullet \mathbf{B} = \mathbf{B} \bullet \mathbf{A}$; $\mathbf{A} \bullet (\mathbf{B} + \mathbf{C}) = \mathbf{A} \bullet \mathbf{B} + \mathbf{A} \bullet \mathbf{C}$

$\mathbf{A} \bullet \mathbf{A} = |\mathbf{A}|^2$

$\mathbf{i} \bullet \mathbf{i} = \mathbf{j} \bullet \mathbf{j} = \mathbf{k} \bullet \mathbf{k} = 1$

$\mathbf{i} \bullet \mathbf{j} = \mathbf{j} \bullet \mathbf{k} = \mathbf{k} \bullet \mathbf{i} = 0$

If $\mathbf{A} \bullet \mathbf{B} = 0$, then either $\mathbf{A} = 0$, $\mathbf{B} = 0$, or **A** is perpendicular to **B**.

$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$

$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) + (\mathbf{A} \times \mathbf{C})$

$(\mathbf{B} + \mathbf{C}) \times \mathbf{A} = (\mathbf{B} \times \mathbf{A}) + (\mathbf{C} \times \mathbf{A})$

$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$

$\mathbf{i} \times \mathbf{j} = \mathbf{k} = -\mathbf{j} \times \mathbf{i}$; $\mathbf{j} \times \mathbf{k} = \mathbf{i} = -\mathbf{k} \times \mathbf{j}$

$\mathbf{k} \times \mathbf{i} = \mathbf{j} = -\mathbf{i} \times \mathbf{k}$

If $\mathbf{A} \times \mathbf{B} = 0$, then either $\mathbf{A} = 0$, $\mathbf{B} = 0$, or **A** is parallel to **B**.

$$\nabla^2 \varphi = \nabla \bullet (\nabla \varphi) = (\nabla \bullet \nabla) \varphi$$

$$\nabla \times \nabla \varphi = 0$$

$$\nabla \bullet (\nabla \times \boldsymbol{A}) = 0$$

$$\nabla \times (\nabla \times \boldsymbol{A}) = \nabla (\nabla \bullet \boldsymbol{A}) - \nabla^2 \boldsymbol{A}$$

Progressions and Series

Arithmetic Progression

To determine whether a given finite sequence of numbers is an arithmetic progression, subtract each number from the following number. If the differences are equal, the series is arithmetic.

1. The first term is a
2. The common difference is d
3. The number of terms is n
4. The last or *n*th term is I
5. The sum of *n* terms is S

$$I = a + (n-1)d$$

$$S = \frac{n(a+I)}{2} = \frac{n[2a + (n-1)d]}{2}$$

Geometric Progression

To determine whether a given finite sequence is a geometric progression (G.P.), divide each number after the first by the preceding number. If the quotients are equal, the series is geometric.

1. The first term is a
2. The common ratio is r
3. The number of terms is n
4. The last or *n*th term is I
5. The sum of *n* terms is S

$$I = ar^{n-1}$$

$$S = \frac{a(1-r^n)}{1-r}; r \neq 1$$

$$S = \frac{a - rI}{1-r}; r \neq 1$$

$$\lim_{n\to\infty} S_n = \frac{a}{1-r}; r < 1$$

A G.P. converges if $|r| < 1$ and it diverges if $|r| \geq 1$

Properties of Series

$$\sum_{i=1}^{n} c = nc;\ c = constant$$

$$\sum_{i=1}^{n} cx_i = c\sum_{i=1}^{n} x_i$$

$$\sum_{i=1}^{n}(x_i + y_i - z_i) = \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} z_i$$

$$\sum_{x=1}^{n} x = \frac{n+n^2}{2}$$

1. A power series in, which is convergent in the interval $-1 < x < 1$ $(or - 1 < x - a < 1)$, defines a function of x which is continuous for all values of x within the interval and is said to represent the function in that interval.
2. A power series may be differentiated term by term, and the resulting series has the same interval of convergence as the original series (except possibly at the end points of the interval).
3. A power series may be integrated term by term provided the limits of integration are within the interval of convergence of the series.

4. Two power series may be added, subtracted, or multiplied, and the resulting series in each case is convergent, at least, in the interval common to the two series.
5. Using the process of long division (as for polynomials), two power series may be divided one by the other.

Taylor's Series

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots$$

is called *Taylor's series*, and the function $f(x)$ is said to be expanded about the point a in a Taylor's series.

If $a = 0$, the Taylor's series equation becomes a *Maclaurin's series*.

Differential Calculus

The Derivative

For any function $y = f(x)$,

The derivative $= D_x y = \frac{dy}{dx} = y'$

$$y' = \lim_{\Delta x \to 0}\left[\frac{\Delta y}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left\{\frac{[f(x+\Delta x) - f(x)]}{\Delta x}\right\}$$

= the slope of the curve $f(x)$

Test for a Maximum

$y = f(x)$ is a maximum for $x = a$,

if $f'(a) = 0$ and $f''(a) < 0$

Test for a Minimum

$y = f(x)$ is a minimum for $x = a$,

if $f'(a) = 0$ and $f''(a) > 0$

Test for a Point of Inflection

$y = f(x)$ has a point of inflection at $x = a$,

if $f''(a) = 0$, and

if $f''(x)$ changes sign as x increases through $x = a$.

The Partial Derivative

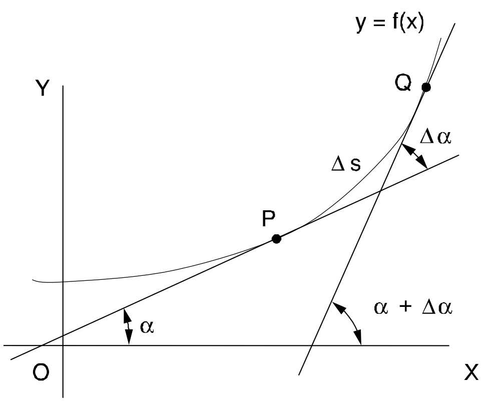
In a function of two independent variables x and y, a derivative with respect to one of the variables may be found if the other variable is *assumed* to remain constant. If y *is kept fixed*, the function

$$z = f(x, y)$$

becomes a function of the *single variable* x, and its derivative (if it exists) can be found. This derivative is called the *partial derivative of* z *with respect to* x. The partial derivative with respect to x is denoted as follows:

$$\frac{dz}{dx} = \frac{df(x, y)}{dx}$$

The Curvature of Any Curve



Wade, Thomas L., *Calculus*, Copyright © 1953 by Ginn & Company.

The curvature $K$of a curve at P is the limit of its average curvature for the arc PQ as Q approaches P. This is also expressed as: the curvature of a curve at a given point is the rate-of-change of its inclination with respect to its arc length.

$$K = \operatorname*{limit}_{\Delta s \to 0} \frac{\Delta \alpha}{\Delta s} = \frac{d\alpha}{ds}$$

Curvature in Rectangular Coordinates

$$K = \frac{y"}{[1 + (y')^2]^{\frac{3}{2}}}$$

When it may be easier to differentiate the function with respect to y rather than x, the notation x' will be used for the derivative.

$$x' = \frac{dx}{dy}$$

$$K = \frac{-x''}{[1 + (x')^2]^{\frac{3}{2}}}$$

The Radius of Curvature

The *radius of curvature* R at any point on a curve is defined as the absolute value of the reciprocal of the curvature K at that point.

$$R = \frac{1}{|K|} \qquad (K \neq 0)$$

$$R = \left| \frac{\left[1+(y')^2\right]^{\frac{3}{2}}}{|y''|} \right| \qquad (y'' \neq 0)$$

Integral Calculus

The definite integral is defined as:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x_i = \int_a^b f(x)dx$$

Also, $\Delta x_i \to 0$ for all i.

A table of derivatives and integrals is also available in this reference document. The integral equations can be used along with the following methods of integration:

A. Integration by Parts (integral equation #6),
B. Integration by Substitution, and
C. Separation of Rational Fractions into Partial Fractions.

Derivatives and Indefinite Integrals

In these formulas, *u*, *v*, and *w* represent functions of *x*. Also, *a*, *c*, and *n* represent constants. All arguments of the trigonometric functions are in radians. A constant of integration should be added to the integrals. To avoid terminology difficulty, the following definitions are followed: arcsin *u* = $\sin^{-1} u$, $(\sin u)^{-1} = 1/\sin u$.

Derivatives

1. $\frac{dc}{dx} = 0$
2. $\frac{dx}{dx} = 1$
3. $\frac{d(cu)}{dx} = c\frac{du}{dx}$
4. $\frac{d(u+v-w)}{dx} = \frac{du}{dx} + \frac{dv}{dx} - \frac{dw}{dx}$
5. $\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
6. $\frac{d(uvw)}{dx} = uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}$
7. $\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
8. $\frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx}$
9. $\frac{d[f(u)]}{dx} = \left\{\frac{d[f(u)]}{du}\right\}\frac{du}{dx}$
10. $\frac{du}{dx} = \frac{1}{\frac{dx}{du}}$
11. $\frac{d(\log_a u)}{dx} = (\log_a e)\frac{1}{u}\frac{du}{dx}$
12. $\frac{d(\ln u)}{dx} = \frac{1}{u}\frac{du}{dx}$
13. $\frac{d(a^u)}{dx} = (\ln a)\, a^u \frac{du}{dx}$
14. $\frac{d(e^u)}{dx} = e^u \frac{du}{dx}$
15. $\frac{d(u^v)}{dx} = vu^{v-1}\frac{du}{dx} + (\ln u)u^v\frac{dv}{dx}$
16. $\frac{d(\sin u)}{dx} = \cos u\frac{du}{dx}$
17. $\frac{d(\cos u)}{dx} = -\sin u\frac{du}{dx}$
18. $\frac{d(\tan u)}{dx} = \sec^2 u\frac{du}{dx}$
19. $\frac{d(\cot u)}{dx} = -\csc^2 u\frac{du}{dx}$
20. $\frac{d(\sec u)}{dx} = \sec u \tan u\frac{du}{dx}$
21. $\frac{d(\csc u)}{dx} = -\csc u \cot u\frac{du}{dx}$
22. $\frac{d(\sin^{-1} u)}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$ $\quad \left(-\frac{\pi}{2} \le \sin^{-1} u \le \frac{\pi}{2}\right)$
23. $\frac{d(\cos^{-1} u)}{dx} = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$ $\quad (0 \le \cos^{-1} u \le \pi)$
24. $\frac{d(\tan^{-1} u)}{dx} = \frac{1}{1+u^2}\frac{du}{dx}$ $\quad \left(-\frac{\pi}{2} < \tan^{-1} u < \frac{\pi}{2}\right)$
25. $\frac{d(\cot^{-1} u)}{dx} = -\frac{1}{1+u^2}\frac{du}{dx}$ $\quad (0 < \cot^{-1} u < \pi)$

26. $\frac{d(sec^{-1}u)}{dx} = \frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}$ $\left(0 < sec^{-1}u < \frac{\pi}{2}\right)\left(-\pi \leq sec^{-1}u < -\frac{\pi}{2}\right)$

27. $\frac{d(csc^{-1}u)}{dx} = -\frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}$ $\left(0 < csc^{-1}u \leq \frac{\pi}{2}\right)\left(-\pi < csc^{-1}u \leq -\frac{\pi}{2}\right)$

Integrals

1. $\int d\, f(x) = f(x)$
2. $\int dx = x$
3. $\int a\, f(x)\, dx = a \int f(x) dx$
4. $\int [u(x) \pm v(x)] dx = \int u(x) dx \pm \int v(x) dx$
5. $\int x^m dx = \frac{x^{m+1}}{m+1}$ $(m \neq -1)$
6. $\int u(x) dv(x) = u(x)v(x) - \int v(x) du(x)$
7. $\int \frac{dx}{ax+b} = \frac{1}{a} ln|ax+b|$
8. $\int \frac{d(x)}{\sqrt{x}} = 2\sqrt{x}$
9. $\int a^x dx = \frac{a^x}{lna}$
10. $\int \sin x\, dx = -\cos x$
11. $\int \cos x\, dx = \sin x$
12. $\int \sin^2 x dx = \frac{x}{2} - \frac{\sin 2x}{4}$
13. $\int cos^2\, x dx = \frac{x}{2} + \frac{\sin 2x}{4}$
14. $\int x \sin x\, dx = \sin x - x \cos x$
15. $\int x \cos x\, dx = \cos x + x \sin x$
16. $\int \sin x \cos x\, dx = \frac{sin^2 x}{2}$
17. $\int \sin ax \cos bx\, dx = -\frac{\cos(a-b)x}{2(a-b)} - \frac{\cos(a+b)x}{2(a+b)} (a^2 \neq b^2)$
18. $\int \tan x\, dx = -ln|\cos x| = ln\, |\sec x|$
19. $\int \cot x\, dx = -ln|\csc x| = ln\, |\sin x|$
20. $\int \tan^2 x\, dx = \tan x - x$
21. $\int \cot^2 x\, dx = -\cot x - x$
22. $\int e^{ax}\, dx = \left(\frac{1}{a}\right) e^{ax}$
23. $\int xe^{ax}\, dx = \left(\frac{e^{ax}}{a^2}\right)(ax - 1)$
24. $\int \ln x\, dx = x\, [\ln(x) - 1]$ $(x > 0)$
25. $\int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$ $(a \neq 0)$
26. $\int \frac{dx}{ax^2+c} = \frac{1}{\sqrt{ac}} \tan^{-1}(x\sqrt{\frac{a}{c}}$ $(a > 0, c > 0)$
27. $\int \frac{dx}{ax^2+bx+c} = \frac{2}{\sqrt{4ac-b^2}} \tan^{-1} \frac{2ax+b}{\sqrt{4ac-b^2}}$ $(4ac - b^2 > 0)$
28. $\int \frac{dx}{ax^2+bx+c} = \frac{1}{\sqrt{b^2-4ac}} \ln \frac{|2ax+b-\sqrt{b^2-4ac}|}{|2ax+b+\sqrt{b^2-4ac}|}$ $(b^2 - 4ac > 0)$
29. $\int \frac{dx}{ax^2+bx+c} = -\frac{2}{2ax+b}$ $(b^2 - 4ac = 0)$

Mensuration of Areas and Volumes

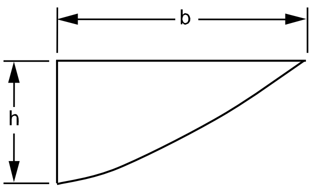
Nomenclature

A = total surface area

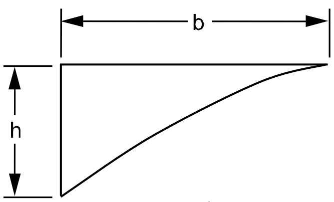
P = perimeter

V = volume

Parabola

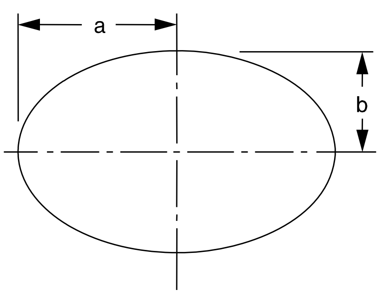


A = 2bh/3



A = bh/3

Ellipse



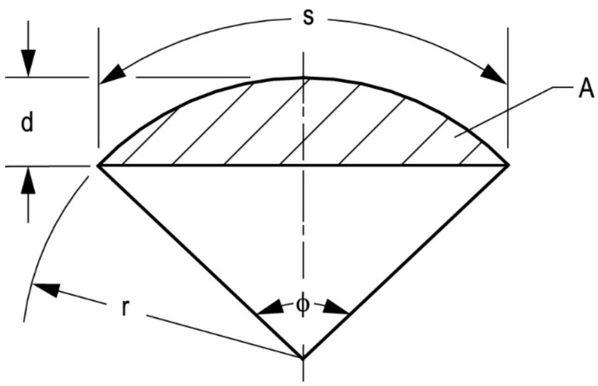
A = πab

$$P_{\text{approx}} = 2\pi\sqrt{\frac{(a^2 + b^2)}{2}}$$

$$P = \pi(a+b)\begin{cases} 1 + (\frac{1}{2})^2\lambda^2 + (\frac{1}{2}\times\frac{1}{4})^2\lambda^4 \\ +(\frac{1}{2}\times\frac{1}{4}\times\frac{3}{6})^2\lambda^6 + (\frac{1}{2}\times\frac{1}{4}\times\frac{3}{6}\times\frac{5}{8})^2\lambda^8 \\ +(\frac{1}{2}\times\frac{1}{4}\times\frac{3}{6}\times\frac{5}{8}\times\frac{7}{10})^2\lambda^{10}+\dots \end{cases}$$

where $\lambda = \frac{a-b}{a+b}$

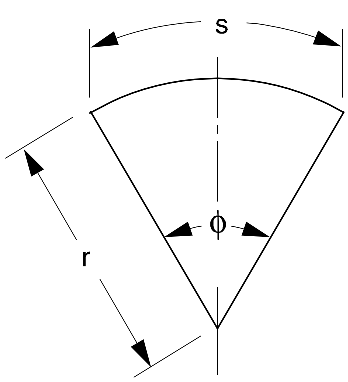
Circular Segment



$$A = \frac{[r^2(\phi - sin\,\phi)]}{2}$$

$$\phi = \frac{s}{r} = 2\left\{\arccos\left[\frac{r-d}{r}\right]\right\}$$

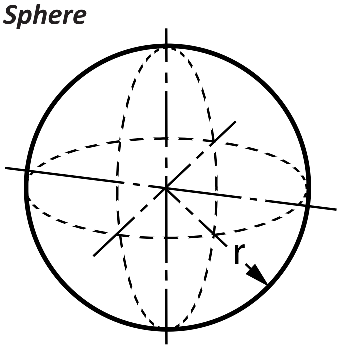
Circular Sector



$$A = \frac{\phi r^2}{2} = \frac{sr}{2}$$

$$\phi = \frac{s}{r}$$

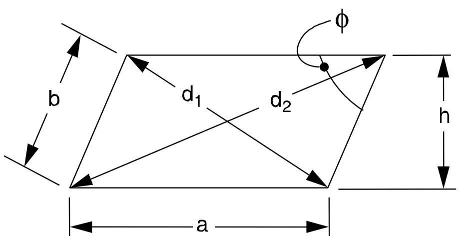
Sphere



$$V = \frac{4\pi r^3}{3} = \frac{\pi d^3}{6}$$

$$A = 4\pi r^2 = \pi d^2$$

Parallelogram



$$P = 2(a + b)$$

$$d_1 = \sqrt{a^2 + b^2 - 2ab(\cos\phi)}$$

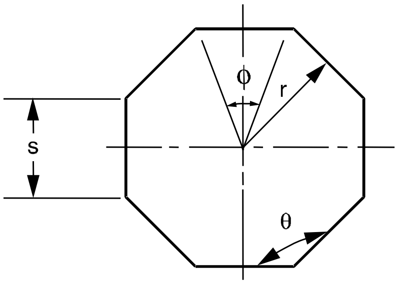
$$d_2 = \sqrt{a^2 + b^2 + 2ab(\cos\phi)}$$

$$d_1{}^2 + d_2{}^2 = 2(a^2 + b^2)$$

$$A = ah = ab(\sin\phi)$$

If $a = b$, the parallelogram is a rhombus.

Regular Polygon (n equal sides)



$$\phi = \frac{2\pi}{n}$$

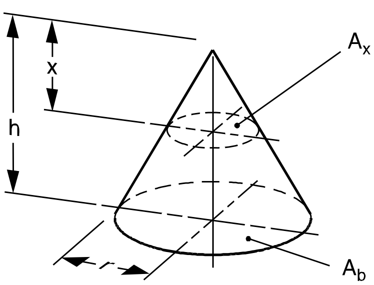
$$\theta = \left[\frac{\pi(n-2)}{n}\right] = \pi\left(1 - \frac{2}{n}\right)$$

$$P = ns$$

$$s = 2r\left[\tan\left(\frac{\phi}{2}\right)\right]$$

$$A = \frac{nsr}{2}$$

Right Circular Cone



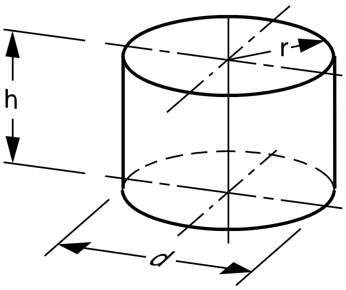
$$V = \frac{\pi r^2 h}{3}$$

$$A = side\ area + base\ a$$

$$A = \pi r\left(r + \sqrt{r^2 + h^2}\right)$$

$$A_x : A_b = x^2 : h^2$$

Right Circular Cylinder

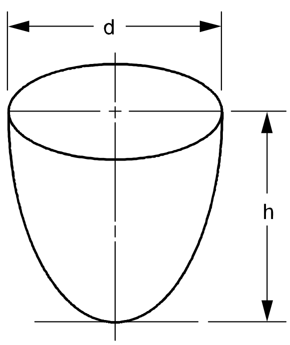


$$V = \pi r^2 h = \frac{\pi d^2 h}{4}$$

$$A = side\ area + end\ areas = 2\pi r(h + r)$$

Gieck, K. & R. Gieck, Engineering Formulas, 6th Ed., Copyright © 1967 by Gieck Publishing.

Paraboloid of Revolution



$$V = \frac{\pi d^2 h}{8}$$

CENTROIDS OF MASSES, AREAS, LENGTH, AND VOLUMES

The following formulas are for discrete masses, areas, lengths, and volumes:

$\mathbf{r}_c = \Sigma\, m_n \mathbf{r}_n / \Sigma\, m_n$

where

m_n = the *mass of each particle* making up the system,
$\mathbf{r}_n$ = the *radius vector* to each particle from a selected reference point, and
$\mathbf{r}_c$ = the *radius vector* to the *centroid of the total mass* from the selected reference point.

The *moment of area* (M_a) is defined as

$M_{ay} = \Sigma\, x_n a_n$

$M_{ax} = \Sigma\, y_n a_n$

The *centroid of area* is defined as

$x_{ac} = M_{ay} /A = \Sigma x_n a_n/A$

$y_{ac} = M_{ax} /A = \Sigma y_n a_n/A$

where $A = \Sigma a_n$

Centroids and Moments of Inertia

The *location of the centroid of an area*, bounded by the axes and the function $y = f(x)$, can be found by integration.

$$x_c = \int \frac{xdA}{A}$$

$$y_c = \int \frac{ydA}{A}$$

$$A = \int f(x)dx$$

$$dA = f(x)dx = g(y)dy$$

The *first moment of area* with respect to the *y*-axis and the *x*-axis, respectively, are:

$$M_y = \int x\, dA = x_c A$$

$$M_x = \int y\, dA = y_c A$$

When either $\bar{x}$ or $\bar{y}$ is of finite dimensions,

then $\int x\, dA$ or $\int y\, dA$ refer to the centroid x or y of dA in these integrals. The *moment of inertia* (*second moment of area*) with respect to the *y*-axis and the *x*-axis, respectively, are:

$$Iy = \int x2\, dA$$

$$Ix = \int y2\, dA$$

The moment of inertia taken with respect to an axis passing through the area's centroid is the *centroidal moment of inertia*. The *parallel axis theorem* for the moment of inertia with respect to another axis parallel with and located *d* units from the centroidal axis is expressed by

$$I_{\text{parallel axis}} = I_c + Ad^2$$

In a plane, $J = \int r^2 dA = I_x + I_y$